# MONTHLY NOTICES

OF THE

# ROYAL ASTRONOMICAL SOCIETY.

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Dr. A. C. D. CROMMELIN, PRESIDENT, in the Chair.

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The election by the Council of the following Fellows was duly confirmed :—

William Elijah Anderson, Ph.D., Professor of Mathematics, Miami University, Oxford, Ohio, U.S.A. (proposed by J. J. Nassau);

Gorakh Prasad, D.Sc., Reader in Mathematics, University of Allahabad, India (proposed by R. A. Sampson);

Ralph Leslie Robinson, B.A., M.R.C.S., L.R.C.P., 30 The Parks, Minehead, Somerset (proposed by A. M. Newbegin); and

Thomas T. Whitehead, 2 Valley Terrace, Simonstone, near Burnley, Lancs (proposed by S. Chapman).

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The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

Ernest Agar Beet, B.Sc., Epworth College, Rhyl, Flintshire (proposed by E. H. Neville);

John Morton Billing, Public Works Dept., Johore Bahru, Malay States (proposed by J. Jackson);

Francis Gilbert Brown, 13 Park Hill Gardens, Sydenham, London, S.E. 26 (proposed by H. H. Turner);

Commander Evan P. Cameron, R.D., R.N.R., 129 Kenilworth Court, Putney, London, S.W. (proposed by A. D. Ross);

Jacques François Jean Gérard Cox, D.Sc., Assistant Professor, Brussels University, 139 Avenue Buyl, Brussels (proposed by C. Parvulesco);

Robert Tait Duthie, Alexandra Terrace, Fraserburgh, Aberdeenshire (proposed by W. H. Coombs) ;

Sir Patrick Fagan, K.C.I.E., 73 Canfield Gardens, London, N.W. (proposed by H. F. Baker) ;

Emrys Howells, M.Sc., The Technical College, Cardiff, and 4 Llandough Street, Cardiff, S. Wales (proposed by A. Buxton) ;

Anton Gotfred Hoyer, Bwana Mkuba, N. Rhodesia (proposed by D. L. Forbes) ;

Diwan Ramchandra Kapur, M.A.S.B., F.R.S.A., Diwan Balmokund Kapur Lane, Benares City, U.P., India (proposed by A. S. Ganesan) ;

James Duncan McNeile, M.A., Wellington College, Berks (proposed by T. E. R. Phillips) ;

Louis Rosenhead, Ph.D., St. John's College, Cambridge (proposed by H. Jeffreys) ;

William Valentine Shearer, B.Sc., Whitworth Scholar, A.M.I.C.E., M.I.M.E., City and Guilds (Engineering) College, Imperial College of Science and Technology, London, S.W. 7 (proposed by H. Knox-Shaw) ;

Jacques Simard, B.A., M.B.A., 1566 St. Denis Street, Montreal, Canada (proposed by T. L. Gillett) ;

René Simard, S.B., S.M., 1566 St. Denis Street, Montreal, Canada (proposed by T. L. Gillett) ;

Charles H. Smiley, M.A., Ph.D., 18 Kidbrook Park Road, London, S.E. 3 (proposed by L. J. Comrie) ;

Robert Gidner Spencer, D.L.C., The School House, Gembling, E. Yorks (proposed by E. Harrison) ;

Cyril Longford Twentyman, 21 Bathurst Road, Grassendale, Liverpool (proposed by E. Hawks) ;

James Ernest Grant Yalden, 120 Woodridge Place, Leonia, N.J., U.S.A. (proposed by D. B. Pickering) ;

George Patrick Meredith, M.Sc., 13 Elmwood, Welwyn Garden City, Herts (proposed by R. Ballard) ; and

Jacob Werner, Imperial Hotel, Piet Retief, Transvaal, S. Africa (proposed by D. L. Forbes).

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Two hundred and four presents were announced as having been received since the last meeting, including, amongst others :—

Catania, R. Osservatorio, Catalogo Astrofotografico, 1900·0, vol. 6, pt. 2 (fasc. 42).

E. Chappell, A Table of Coefficients to facilitate Interpolation by means of the Formulæ of Gauss, Bessel, and Everett.

Danske Magazin, Bind 1, 2 (1745–46) ; Tyge Brahe, en historisk Fremstilling efter-trykte og utrykte Kilder ; Urania, Aarbog for 1846 (presented by Dr. W. Doberck).

Greenwich, Royal Observatory, Astrographic Catalogue, vol. 5.

Ebn Iounis, Le livre de la Grande Table Hakémite (presented by Dr. E. B. Knobel).

Königsberg, Universitäts-Sternwarte, Astronomische Beobachtungen, Abt. 45.

L. Milne-Thomson, Standard Table of Square Roots.

Sydney Observatory, Astrographic Catalogue, vols. 5 and 6.

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*Celestial Photographs.*

The following have been added to the list of Celestial Photographs reproduced by the Society for sale to the Fellows :—

R.A.S. Ref. No.

413. Eruptive Prominence, 1928 Nov. 19. (Maximum height, 567,000 miles.) Kodaikanal.

414. Solar Corona, 1929 May 9. Alor Star. Exp. 20 sec. (from $100^s$ to $120^s$). 6-in. Refr., F.L. 45 ft. Jackson.

415. Solar Corona, 1929 May 9. Alor Star. Exp. 10 sec. (from $200^s$ to $210^s$). 6-in. Refr., F.L. 45 ft. Jackson.

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*The Planetary Theory with the True Longitude as Independent Variable.*
By Ernest W. Brown, Sc.D., F.R.S.

1. Laplace, and following him Plana, used the equations of motion with the projection of the true longitude on the plane of reference as independent variable for extensive application to an actual problem—the theory of the motion of the moon. In my theory of the Trojan group of asteroids * I used the true longitude reckoned in the manner usual in the planetary theory, namely, to the node along the plane of reference and then along the osculating orbit. The work might have been simplified by the use of the independent variable adopted here, namely, the true longitude reckoned wholly in the osculating plane, that is, from a " departure point " in that plane. This last variable has been used by Gyldén for planetary developments.

This variable has several advantages. In the first place the equations of motion become almost as simple as those for the canonical elements. Secondly, the three required " co-ordinates " are found directly from these equations. Thirdly, the effect of the motion of the osculating plane on the motion within that plane appears to be smaller than with any other variable. Their most important property from the point of view of the calculation consists in the fact that the equations for the inverse of the radius vector and for the sine of the latitude have the form $(D^2 + 1)x = a + p$, where $a$ is a constant and $p$ has the disturbing mass as a factor so that they are integrated without tedious

* *Trans. Yale Univ. Obs.*, vol. iii. pts. 1, 3 (1926).

multiplications of series. An auxiliary variable $q$ is obtained by integrating an equation of the form $Dq = p'$, and finally, $t$ is found from the equation $Dt = q^{\frac{1}{2}}u^{-2}$. The chief disadvantage is the somewhat greater complexity in the development of the disturbing forces, due to the fact that the co-ordinates of the disturbing planet have to be expressed in terms of the true longitude of the disturbed planet. This disadvantage disappears in a numerical development, and a literal development does not present any very grave complications.

In dealing with the terms of long period it is often of advantage to use the equations for the variations of the elements, and it is shown below that these are obtained and integrated without difficulty.

The present paper is mainly concerned with the application of the method to planetary or asteroid problems. The first approximation along powers of the disturbing mass is developed with detailed explanations sufficient to show the plan of procedure and the determination of the arbitrary constants of the solution. In connection with the second approximation, it is shown how the principal long-period terms which arise therein can be obtained at once from the first approximation. The formulæ, which I believe are new, are applied to the mutual perturbations of Jupiter and Saturn, and by comparison with Hill's numerical theory are shown to give a high degree of accuracy for certain terms of very long period which arise from the first approximation to the great inequality in the motions of those bodies.

Those familiar with Hansen's method will recognize some similarity. Substantially the perturbations are added to the mean anomaly as in Hansen's work, but they are expressed in terms of the true longitude instead of in terms of the time. The use of this variable eliminates the necessity for the introduction of Hansen's variables $\bar{r}, \bar{f}, \ldots$ and thus avoids the multiplications of series and the presence of multiples of *three* arguments, later reduced to two. A final section shows that, if it be desired at the end of the work to transform back to $t$ as the independent variable, the change can be made with but little additional calculation.

In order to keep this paper within reasonable limits of space I have assumed that the reader is familiar with the methods and results of the usual perturbation theories. The fundamental ideas involved in the planetary problem are simple: it is in the detailed developments that difficulties arise—chiefly with reference to the determination of the arbitrary constants and of the coefficients of the long-period terms. An effort has therefore been made to explain these matters in some detail, and I am indebted to Professor C. A. Shook for his help in pointing out obscurities and errors in the exposition.

2. Let (fig. 1) CONM be a fixed plane of reference in which C is the origin and CO a fixed line. Let O'NP be the osculating plane of motion of a particle P of mass $m_1$, so that the tangent line to the path of P always lies within this plane. Put

$$\text{OCN} = \theta, \qquad \text{PNM} = i, \qquad \text{NCP} = v - \theta, \qquad \text{O'CP} = \mathrm{v},$$

the point O' (on the unit sphere with C as centre) being so defined that

$dv/dt$ is the angular velocity of CP in the osculating plane : the point O′ is known as a " departure point " and lies on an orthogonal trajectory of the plane. Then

$$\frac{dv}{dt} = \frac{d}{dt}(v - \theta) + \cos i \frac{d\theta}{dt},$$

or, if $\Gamma \equiv 1 - \cos i$,

$$\frac{dv}{dt} = \frac{dv}{dt} - \Gamma\frac{d\theta}{dt}. \quad . \quad . \quad . \quad . \quad (2.1)$$

The components of the velocity of $m_1$, along $r$ the radius vector, perpendicular to $r$ in the osculating plane, and perpendicular to that plane are $\dot{r}$, $r\dot{v}$, 0. The kinetic energy of $m_1$ is therefore given by

$$2T = m_1\left(\frac{dr}{dt}\right)^2 + m_1 r^2\left(\frac{dv}{dt} - \Gamma\frac{d\theta}{dt}\right)^2.$$

This contains four variables and there are only three degrees of

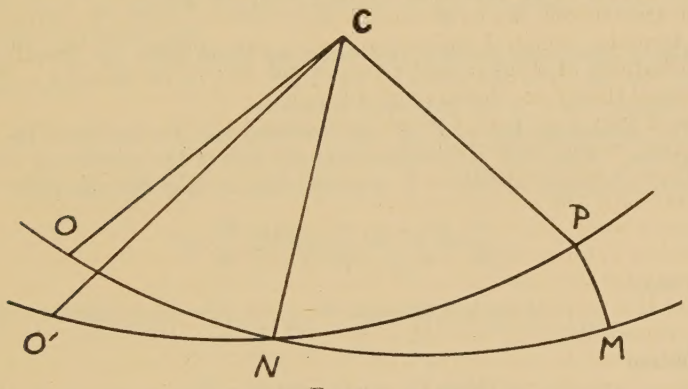


Fig. 1.

freedom. If we imagine a particle of mass $m_2$ compelled by frictionless constraints to remain in the line CN at a constant distance $b$ from C, we must add a term $m_2 b^2\dot{\theta}^2$ to $2T$, and the system will then have four degrees of freedom.

Let the force function

$$m_1\mu\left(\frac{1}{r} + R\right)$$

be supposed to be expressed as a function of $r$, $v$, $\Gamma$, $\theta$, $t$. We can now use Lagrange's equations and obtain, after putting $m_2 = 0$,

$$\frac{d^2r}{dt^2} - r\left(\frac{dv}{dt}\right)^2 = -\frac{\mu}{r^2} + \mu\frac{\partial R}{\partial r}, \qquad \frac{d}{dt}\left(r^2\frac{dv}{dt}\right) = \mu\frac{\partial R}{\partial v}, \quad . \quad (2.2)$$

$$\frac{d}{dt}\left(r^2\Gamma\frac{dv}{dt}\right) = -\mu\frac{\partial R}{\partial\theta}, \qquad r^2\frac{dv}{dt}\frac{d\theta}{dt} = \mu\frac{\partial R}{\partial\Gamma}, \quad . \quad (2.3)$$

where $dv/dt$ is defined by (2.1).

3. Let us now change the independent variable from $t$ to v. If we put

$$\mathrm{D} \equiv \frac{d}{d\mathrm{v}}, \qquad \sqrt{\frac{\mu}{q}} \equiv r^2\frac{d\mathrm{v}}{dt}, \qquad u \equiv \frac{1}{r}, \quad . \quad . \quad (3.1)$$

the equations of motion can be written

$$\mathrm{D}q = -\frac{2q^2}{u^2}\frac{\partial \mathrm{R}}{\partial v}, \qquad (\mathrm{D}^2+1)u - q = q\left(\frac{\partial \mathrm{R}}{\partial u} - \frac{\mathrm{D}u}{u^2}\frac{\partial \mathrm{R}}{\partial v}\right), \quad (3.2)$$

$$\mathrm{D}t = \left(\frac{q}{\mu}\right)^{\frac{1}{2}}\frac{1}{u^2}, \qquad \mathrm{D}v = 1 + \Gamma\mathrm{D}\theta, \quad . \quad . \quad (3.3)$$

$$\mathrm{D}(\Gamma q^{-\frac{1}{2}}) = -\frac{q^{\frac{1}{2}}}{u^2}\frac{\partial \mathrm{R}}{\partial \theta}, \qquad \mathrm{D}\theta = \frac{q}{u^2}\frac{\partial \mathrm{R}}{\partial \Gamma}. \quad . \quad . \quad (3.4)$$

From (3.2) we deduce

$$\tfrac{1}{2}\mathrm{D}\left\{\frac{(\mathrm{D}u)^2+u^2}{q} - 2u\right\} = \frac{\partial \mathrm{R}}{\partial v} + \mathrm{D}u\frac{\partial \mathrm{R}}{\partial u}. \quad . \quad . \quad (3.5)$$

These equations become the same as those used for the Trojan group (*loc. cit.*) if in the latter we put $\Gamma\mathrm{D}\theta = 0$ in the factor $1 - \Gamma\mathrm{D}\theta$, and change $v$ to v in the left-hand members.

The equations for $v$, $\Gamma$, $\theta$ can conveniently be replaced by two equations. Since CP is the instantaneous axis of the osculating plane, the rate of change of PM $\equiv$ L depends only on that of v, so that

$$\frac{d}{dt}(\sin \mathrm{L}) = \frac{\partial}{\partial \mathrm{v}}(\sin \mathrm{L}).\frac{d\mathrm{v}}{dt}.$$

And since

$$\sin \mathrm{L} = \sin i \sin(v-\theta) = \sin i \sin(\mathrm{v} - \int \cos i d\theta), \quad . \quad (3.6)$$

we obtain

$$\mathrm{D}(\sin \mathrm{L}) = \sin i \cos(v-\theta). \quad . \quad . \quad . \quad (3.7)$$

Again, differentiating (3.7), we obtain with the help of (2.1),

$$\mathrm{D}^2(\sin \mathrm{L}) = -\sin i \sin(v-\theta)(1 - \cos i\,\mathrm{D}\theta) + \cos i \cos(v-\theta)\mathrm{D}i.$$

But since $\mathrm{D}(\sin \mathrm{L}) = \partial/\partial \mathrm{v}(\sin \mathrm{L})$, we have

$$\mathrm{D}i \cos i \sin(v-\theta) - \cos i \sin i\,\mathrm{D}\theta \cos(v-\theta) = 0.$$

Whence, eliminating $\mathrm{D}i$ between this and the previous equation, we obtain

$$(\mathrm{D}^2+1)\sin \mathrm{L} = \frac{\sin i \cos i}{\sin(v-\theta)}\mathrm{D}\theta = \frac{\sin i \cos i}{\sin(v-\theta)}\frac{q}{u^2}\frac{\partial R}{\partial \Gamma}. \quad . \quad (3.8)$$

Since $\partial \mathrm{R}/\partial\Gamma$ contains $\sin(v-\theta)$ as a factor, there is no discontinuity in the right-hand member when $\sin(v-\theta) = 0$.

Finally, if we put

$$\mathrm{P} \equiv \frac{\partial \mathrm{R}}{\partial u} - \frac{\mathrm{D}u}{u^2}\frac{\partial \mathrm{R}}{\partial v}, \qquad \mathrm{Q} \equiv -\frac{q}{u^2}\frac{\partial \mathrm{R}}{\partial v}, \qquad \mathrm{Z} \equiv -\frac{q}{u^2 \sin(v-\theta)}\frac{\partial \mathrm{R}}{\partial \Gamma}, \quad (3.9)$$

the equations of motion may be written,

$$Dq = 2qQ, \qquad (D^2 + 1)u = q(1 + P), \qquad Dt = \left(\frac{q}{\mu}\right)^{\frac{1}{2}} \frac{1}{u^2}, \quad (3.10)$$

$$Dv = 1 - (1 - \cos i)Z \sin(v - \theta), \quad (D^2 + 1) \sin L = -(\cos i \sin i)Z; \quad (3.11)$$

the equations (3.4) will only be useful when a second approximation to the portions of $u$, $v$, L which depend on $\Gamma$, $\theta$ is needed.

The immediate object of this paper is to develop the method of finding planetary perturbations by means of (3.10), (3.11).

It may be pointed out that since $Dq$ is of the order of the disturbing forces we can change the dependent variable $u$ to $uf(q)$ without altering the essential characteristics of the method. In particular, we can put $f(q) = q^j$. Transformations with $j = -1, -\frac{1}{4}$ and with some other values of $j$ have been studied in detail. Similar remarks are applicable to the variable sin L. We can also replace the variable $q$ by any function of $q$. But none of these changes seem to produce formulæ as convenient and simple for the purposes of the planetary theory as the variables adopted here.

4. When the disturbing forces P, Q, Z are neglected, the motion is elliptic with

$$\frac{1}{q} = a(1 - e^2), \quad u = q + qe\cos(v - \varpi), \quad \sin L = \sin i \sin(v - \theta), \quad (4.1)$$

with $i$, $\theta$ constant, $v = v$ and with

$$Dt = \left(\frac{a^3}{\mu}\right)^{\frac{1}{2}} \frac{(1 - e^2)^{\frac{3}{2}}}{\{1 + e\cos(v - \varpi)\}^2}. \quad . \quad . \quad (4.2)$$

If we define $n$ by $n^2a^3 = \mu$, the integral of (4.2) is

$$nt + \epsilon = v - E_f, \quad . \quad . \quad . \quad . \quad (4.3)$$

where $E_f$ is the equation of the centre in terms of the true anomaly $v - \varpi$. It may be written in the finite form

$$nt + \epsilon - \varpi = C - e\sin C, \qquad \tan\tfrac{1}{2}C = \left(\frac{1 - e}{1 + e}\right)^{\frac{1}{2}} \tan\tfrac{1}{2}(v - \varpi), \quad (4.4)$$

where C is the eccentric anomaly.

The arbitrary constants are the usual elliptic elements. An additional constant due to the integration of the equation $Dv = 1$ is so determined that $v = v$.

5. The usual method to obtain a first approximation to the perturbations is to substitute the elliptic values of the co-ordinates in the disturbing forces and to solve the equations again. Here P, Q, Z, besides being functions of $u$, $v$, $\Gamma$, $\theta$, are also functions of $r'$, $v'$, the radius vector and longitude of the disturbing body, which for the sake of simplicity of exposition is supposed to move in an ellipse in the plane of reference.

Now $r'/a'$, $v' - \varpi'$ are functions of the eccentricity $e'$ and mean anomaly $g' \equiv n't + \epsilon' - \varpi'$ of the disturbing body, and these have to

be expressed in terms of v by means of the equation (4.3) or (4.4). If we put

$$f \equiv \mathrm{v} - \varpi, \qquad f_1 \equiv \frac{n'}{n}(\mathrm{v} - \epsilon) + \epsilon' - \varpi', \quad . \qquad . \qquad (5.1)$$

we have

$$g' = f_1 - \frac{n'}{n}\mathrm{E}_f = f_1 - \frac{n'}{n}(2e \sin f - \ . \ . \ .), \qquad . \qquad (5.2)$$

by means of which $r'$, $v'$ are expressed in terms of the angles $f, f_1$.

Since $v = \mathrm{v}$ in this approximation, $r$, $v$ are directly expressed in terms of the angle $f$ by means of

$$v = f + \varpi, \qquad 1/r = q + qe \cos f.$$

Thence the forces can be expressed as sums of sines and cosines of multiples of the angles $f$, $f_1$ with coefficients which depend on the longitudes of the perihelia and nodes, on the inclination, the eccentricities and the ratio of the mean distances.

6. Suppose that these series have been obtained. The right-hand members of the equations (3.10, 3.11) will be functions of v only since $\mathrm{D}f = 1$, $\mathrm{D}f_1 = n'/n$. The integral of the equation for $q$ is

$$q = q_0 + \mathrm{D}^{-1}(2q\mathrm{Q}) \equiv q_0 + \delta q, \quad . \qquad . \qquad . \qquad (6.1)$$

where $q_0$ is an arbitrary constant. Substituting this in the equation for $u$, we obtain, on integration,

$$\begin{aligned} u &= q_0 + q_0 e_0 \cos(\mathrm{v} - \varpi_0) + \frac{1}{\mathrm{D}^2 + 1}(\delta q + q_0 \mathrm{P}) \\ &\equiv q_0 + q_0 e_0 \cos(\mathrm{v} - \varpi_0) + \delta u \equiv u_0 + \delta u, \quad . \qquad . \qquad (6.2) \end{aligned}$$

in which $e_0$, $\varpi_0$ are arbitrary constants.

These values for $u$, $q$ are substituted in the equation for $\mathrm{D}t$, neglecting powers of $\delta q$, $\delta u$ higher than the first. We obtain

$$\begin{aligned} \mathrm{D}t &= \left(\frac{q_0}{\mu}\right)^{\frac{1}{2}} \frac{1}{u_0^2}\left(1 + \frac{1}{2}\frac{\delta q}{q_0} - 2\frac{\delta u}{u_0}\right) \\ &= \left(\frac{q_0}{\mu}\right)^{\frac{1}{2}} \frac{1}{u_0^2} + \left(\frac{q_0}{\mu}\right)^{\frac{1}{2}} \frac{1}{u_0^3}\{\tfrac{1}{2}\delta q(1 + e_0 \cos f_0) - 2\delta u\}, \end{aligned} \qquad (6.3)$$

which, after forming the product of the double series in { } by the rapidly converging single series for $1/u_0^3$, can be integrated.

The integral of the first of equations (3.11) gives

$$v = \mathrm{v} - \Gamma_0 \mathrm{D}^{-1}\mathrm{Z} \sin(v - \theta), \quad . \qquad . \qquad (6.4)$$

the additive arbitrary constant being put equal to zero.

Finally the equation (3.11) for $\sin \mathrm{L}$ gives

$$\sin \mathrm{L} = \sin i_0 \sin(\mathrm{v} - \theta_0) - \cos i_0 \sin i_0 \frac{1}{\mathrm{D}^2 + 1}\mathrm{Z}, \quad . \qquad (6.5)$$

in which $i_0$, $\theta_0$ are arbitrary constants.

7. It is necessary to define the arbitrary constants in disturbed motion.

Leaving aside for a moment the definition of $q_0$, consider the expression (6.2) for $u$. There is a constant term in the expansion of $q\mathrm{P}$—call it $(q\mathrm{P})_\mathrm{C}$—but none in $\delta q$. Hence the constant term in $u$ is $q_0 + (q\mathrm{P})_\mathrm{C}$. There is also a term $q_1\mathrm{v}$ in $\delta q$, where $q_1$ is a constant.

There will be terms in $\delta q + q\mathrm{P}$ of the form $\mathrm{A}\cos\mathrm{v} + \mathrm{A}'\sin\mathrm{v}$. The effect of the operator $(\mathrm{D}^2 + 1)^{-1}$ on these is to produce terms

$$-\tfrac{1}{2}\,\mathrm{A}\mathrm{v}\sin\mathrm{v} + \tfrac{1}{2}\,\mathrm{A}'\mathrm{v}\cos\mathrm{v}, \qquad . \qquad . \qquad . \qquad (7.1)$$

together with an arbitrary addition $\alpha\cos\mathrm{v} + \beta\sin\mathrm{v}$. It is convenient to put this latter portion zero, so that the constant portions of the coefficients of $\cos\mathrm{v}$, $\sin\mathrm{v}$ in $u$ shall be represented by $q_0e_0\cos\varpi_0$, $q_0e_0\sin\varpi_0$. The term $q_1\mathrm{v}$ in $\delta q$ produces $q_1\mathrm{v}$ in $u$.

In the equation (6.3) for $\mathrm{D}t$, the first term will give a constant term $(q_0 - q_0e_0^2)^{-\frac{3}{2}}\mu^{-\frac{1}{2}}$ and the other terms will give additional portions. On integration these give a term in $t$ of the form $\kappa\mathrm{v}$. If $(\Gamma\mathrm{D}\theta)_\mathrm{C}$ be the constant term of the right-hand member of the first of equations (3.11) we have $v = \mathrm{v} + \mathrm{v}(\Gamma\mathrm{D}\theta)_\mathrm{C} +$ periodic terms, so that to this order of accuracy

$$\mathrm{v} = v - v(\Gamma\mathrm{D}\theta)_\mathrm{C} + \text{periodic terms.}$$

Hence when $t$ is expressed in terms of $v$, the coefficient of $v$ is $\kappa - \kappa(\Gamma\mathrm{D}\theta)_\mathrm{C}$. This quantity is denoted by $1/n$ where $n$ is the mean motion in the disturbed orbit. As $q_0$ is contained in $\kappa$, this defines $q_0$.

A well-known proposition states that when $t$ is the independent variable there are no terms of the form $\kappa t$ in the mean distance $a$ so long as we confine the work to perturbations of the first order of the masses, and it follows that the longitude contains no term having $t^2$ as a factor. The former statement is still true when we include perturbations of the second order of the masses. It will be shown that to the first order a similar property holds when v is the independent variable:—There are no terms in $a$ of the form $\kappa\mathrm{v}$ and none in $t$ depending on $\mathrm{v}^2$.

There will be terms of the form $\mathrm{v}\sin if$, $\mathrm{v}\cos if$ which are integrated in the usual way, giving rise to terms in $t$ of similar form together with terms of the form $\alpha\sin if$, $\alpha\cos if$ ($\alpha$ constant), and in particular to terms for which $i = 1$. As we have already defined the eccentricity and longitude of perihelion by means of the term with argument $f$ in $u$, that in $t$ with the same argument will differ slightly from its elliptic value: the same thing happens when $t$ is the independent variable.

In a numerical solution in which the mean elements are supposed to be known it is convenient to put

$$q_0 = \frac{1}{a(1 - e^2)} + \delta q_0, \qquad a^3n^2 = \mu, \qquad e_0 = e, \qquad \varpi_0 = \varpi,$$

where $n$, $a$, $e$, $\varpi$ are the given elements. We then carry $\delta q_0$ as an unknown until $t$ is expressed in terms of $v$, when it is determined in the manner just shown. In any case there are small corrections to the elements to be made when the theory is compared with observation: we have been here merely concerned with the relation between $q_0$, $n$.

The definitions of $i_0$, $\theta_0$ are made in a manner similar to those of $e_0$,

$\varpi_0$, namely that $\sin i_0 \cos \theta_0$, $-\sin i_0 \sin \theta_0$ shall be the constant parts of the coefficients of sin v, cos v in sin L when the motion is disturbed. These may be conveniently taken as the mean observed elements in calculating the perturbations.

8. After the forces P, Q, Z have been expressed in terms of $f, f_1$, the numerical work in the operations described above is but little more than dividing a coefficient $A_\lambda$ of $\cos \lambda v$ or $\sin \lambda v$ by $\lambda$ or by $1 - \lambda^2$ where $\lambda$ is of the form $i \pm i'n'/n$, $i$, $i'$ being integers. The only multiplication of two series is that contained in equation (6.3). The main part of the numerical work is therefore the expansion of P, Q, Z in terms of the arguments $f, f_1$.

*Variation of the Elements.*

9. The linear forms of the equations in § 3 suggest that we can at once replace them by equations which may be regarded as those for the variation of the elements. For the discussion of the second approximation, however, more details are necessary.

Put

$$u = q + qe \cos (v - \varpi), \qquad 1/q = a(1 - e^2), \quad . \quad (9.1)$$

and replace the two variables $u$, $q$ by the three $a$, $e$, $\varpi$, choosing as the required additional relation,

$$Du = \frac{\partial u}{\partial v} = -qe \sin (v - \varpi), \quad . \quad . \quad . \quad (9.2)$$

Let us put

$$D_1 \equiv Da\frac{\partial}{\partial a} + De\frac{\partial}{\partial e} + D\varpi\frac{\partial}{\partial \varpi}, \quad . \quad . \quad . \quad (9.3)$$

so that (9.2) may be written $D_1 u = 0$. Differentiating (9.2) and using (9.1) we obtain

$$D^2u + u - q = \frac{Dq}{q} . Du - qD_1\{e \sin (v - \varpi)\}.$$

Whence, with the use of (3.9), (3.10),

$$D_1\{e \sin (v - \varpi)\} = -\frac{\partial R}{\partial u} + \frac{1}{2}\frac{Dq}{q^2}Du = -\frac{\partial R}{\partial u} - \frac{Du}{u^2} . \frac{\partial R}{\partial v}. \quad (9.4)$$

Again substituting (9.1, 2) in (3.5) we obtain

$$D\left(-\frac{1}{2a}\right) = \frac{\partial R}{\partial v} + \frac{\partial R}{\partial u} . Du, \quad . \quad . \quad . \quad (9.5)$$

which determines the variation of $a$.

Next, the equation $D_1 u = 0$ gives, since $D_1 q = Dq$,

$$\{1 + e \cos (v - \varpi)\}Dq + qD_1\{e \cos (v - \varpi)\} = 0,$$

or, from (3.2),

$$D_1\{e \cos (v - \varpi)\} = -\frac{u}{q^2}Dq = \frac{2}{u}\frac{\partial R}{\partial v}. \quad . \quad . \quad (9.6)$$

The equations (9.4), (9.6) determine the variations of $e$, $\varpi$ or of $e\cos\varpi$, $e\sin\varpi$. The variations of $\Gamma$, $\theta$ are given by (3.4) combined with (3.2).

Finally

$$\mathrm{D}t = \frac{q^{\frac{1}{2}}}{\mu^{\frac{1}{2}}u^2} = \left(\frac{a^3}{\mu}\right)^{\frac{1}{2}} \frac{(1-e^2)^{\frac{3}{2}}}{\{1+e\cos(\mathrm{v}-\varpi)\}^2}. \quad . \quad . \quad (9.7)$$

Let us define $n$ in the usual manner, namely, by means of the equation $n^2a^3 = \mu$. The constant term in the expansion of (9.7) is then $1/n$ when the motion is elliptic. Let us put $\mathrm{E}_f$ for the equation of the centre in terms of the true anomaly $\mathrm{v} - \varpi$. Then, in disturbed motion, we have

$$\mathrm{D}t = \frac{1}{n} - \frac{\partial}{\partial \mathrm{v}}\left(\frac{\mathrm{E}_f}{n}\right) = \frac{1}{n} - \mathrm{D}\left(\frac{\mathrm{E}_f}{n}\right) + \mathrm{D}_1\left(\frac{\mathrm{E}_f}{n}\right).$$

Integrating, we obtain

$$t = \int \frac{1}{n} d\mathrm{v} - \frac{\mathrm{E}_f}{n} + \int \mathrm{D}_1\left(\frac{\mathrm{E}_f}{n}\right) d\mathrm{v}. \quad . \quad . \quad (9.8)$$

$$\equiv t_0 - \frac{1}{n}\mathrm{E}_f + t_1,$$

which defines $t_0$, $t_1$. We therefore have

$$\mathrm{D}t_1 = \frac{\mathrm{D}e}{n}\frac{\partial}{\partial e}\mathrm{E}_f + \frac{\mathrm{D}\varpi}{n}\frac{\partial}{\partial \varpi}\mathrm{E}_f + \frac{3}{2}\mathrm{E}_f\left(\frac{a}{\mu}\right)^{\frac{1}{2}}\mathrm{D}a, \quad . \quad . \quad (9.9)$$

from which, with the help of (9.4, 5, 6), $t_1$ may be obtained. The equations (9.8, 9) are formed only for a theoretical discussion made below: in an actual calculation the computed disturbed values of $a$, $e$, $\varpi$ would be substituted in (9.7) and the equation then integrated.

10. Since R is a function of $u$, $v$, $\Gamma$, $\theta$, and of $t$ through its presence in the co-ordinates $r'$, $v'$ of the disturbing planet, we have

$$\mathrm{D}\mathrm{R} = \frac{\partial \mathrm{R}}{\partial v}\mathrm{D}v + \frac{\partial \mathrm{R}}{\partial u}\mathrm{D}u + \frac{\partial \mathrm{R}}{\partial \Gamma}\mathrm{D}\Gamma + \frac{\partial \mathrm{R}}{\partial \theta}\mathrm{D}\theta + \frac{\partial \mathrm{R}}{\partial t}\mathrm{D}t.$$

Substituting for $\mathrm{D}v$, $\mathrm{D}\Gamma$, $\mathrm{D}\theta$ from (3.3, 4) we obtain

$$\mathrm{D}\mathrm{R} = \frac{\partial \mathrm{R}}{\partial v} + \frac{\partial \mathrm{R}}{\partial u}\mathrm{D}u + \frac{\partial \mathrm{R}}{\partial t}\mathrm{D}t. \quad . \quad . \quad . \quad (10.1)$$

This result might also have been obtained from the fact that R depends only on the positions of the two planets, and therefore that its variations depend only on those of v, $t$.

When R has been finally expressed in terms of the elements and the two angles $f, f_1$, equations (5.1, 2) show that

$$\frac{\partial \mathrm{R}}{\partial t} = n' \frac{\partial \mathrm{R}}{\partial f_1}. \quad . \quad . \quad . \quad . \quad (10.2)$$

The equation (9.5), by the use of (10.1, 2), may then be written

$$\mathrm{D}\left(\frac{1}{2a} + \mathrm{R}\right) = n' \frac{\partial \mathrm{R}}{\partial f_1} \mathrm{D}t. \quad . \quad . \quad . \quad (10.3)$$

In a first approximation to the disturbed elements we substitute elliptic values in R and in the right member of (10.3). Since D$t$ does not then contain the angle $f_1$, the expanded right-hand member has no term independent of the arguments $f$, $f_1$. Hence, as when $t$ is the independent variable, $1/a$ contains no secular term. The same result is evidently true to the second order of the disturbing forces when two different disturbing planets are involved, for then D$t$ and the elements are functions of a different angle $f_1'$ belonging to the second disturbing planet. But it does not seem to be a fact that, to the order of the square of the mass of a single disturbing planet, there is no secular term in $1/a$ as is the case when $t$ is the independent variable.

11. The chief difficulties in all planetary work arise through the long-period terms. In the first approximation the equations of § 8 show that if $s\mathrm{v} + s_1$ be the argument of a long-period term, so that $s$ is small compared with unity, the disturbed values of $a$, $e$, $\varpi$, $\Gamma$, $\theta$, $i_1$ contain only the first power of $s$ as a divisor, and that the second power arises in $t$ only through the quadrature $\int 1/n d\mathrm{v}$. But the long-period inequalities in $e$, $\varpi$, $i_1$ will in general have coefficients of the order $1/e$ as compared with the corresponding coefficient in $1/a$.

Hence if $m\mathrm{A}$ be the coefficient of a long-period term in R with a given argument $s\mathrm{v} + s_1$, there will be a long-period term in $t$ with a coefficient of order $m\mathrm{A}/s^2$, and a short-period term of argument $s\mathrm{v} + s_2 \pm \mathrm{v}$ with a coefficient of order $m\mathrm{A}/es$ when A contains $e$ as a factor: if $e$ is not present in A, there will be no associated short-period term in $t$, because $\delta e$, $\delta\varpi$ are then zero. This latter fact ensures the absence of $e$ as a divisor in the co-ordinates and in $t$, just as it does in the co-ordinates when $t$ is the independent variable.

When $e$ is small the associated short-period term may have a coefficient comparable with that of the long-period term. For example, in the action of Jupiter on Saturn the term with the long-period argument $5g' - 2g$ (Hill's notation) has a coefficient $2906''$ in longitude, and the principal associated short-period term, that with argument $4g' - 2g$, has a coefficient $684''$. The effects of these latter terms have to be considered when we proceed to a second approximation.

### *The Second Approximation.*

12. In the great majority of planetary problems, a second approximation is needed only on account of the presence of long-period terms with large coefficients in the first approximation. In order to obtain this second approximation the disturbed values of the elements are supposed to be substituted in the disturbing forces instead of constant values. If we expand by Taylor's theorem, the additional portions of the equations of motion are evidently second derivatives of R multiplied by $\delta a$, $\delta e$, . . ., or by $\delta u$, $\delta t$, . . ., the latter being the portion given by

the first approximation. In these second derivatives we evidently substitute constant elliptic values. Thus we have to deal with products of series, and it is supposed that the product of a sine or a cosine by a sine or cosine is expressed as a sum of sines or cosines of the sum and difference of the angles.

A secular term in an element will produce terms of the forms $\kappa v$ or $\kappa v \cos(sv + s_1)$, in the linear equations for the elements. On integration, the former produces $\frac{1}{2}\kappa v^2$ and the latter

$$\int \kappa v \cos(sv + s_1) dv = \frac{\kappa v}{s} \sin(sv + s_1) + \frac{\kappa}{s^2} \cos(sv + s_1). \quad (12.1)$$

Hence a single integration of such a term can produce a divisor $s^2$ and a double integration a divisor $s^3$.

In the second approximation various powers of $s$, up to $s^4$, appear as divisors. It is the purpose of this and of the following paragraphs to show how these divisors arise and thence to show that with but little additional numerical work we can compute the principal portions of the second approximation with possibly the degree of accuracy needed, and in any case with an approximation which will save much labour.

13. Since short-period terms can only arise in a product when one member of the product contains a short-period term, and since the integration of such terms produces no small divisor, such second-order terms as are of short period can only be of the order of the first-order terms multiplied by the disturbing mass. In general, such terms may be neglected.

Long-period terms formed from the product of two short-period terms are at most of the order $m^2/s^2$ or $m^2/es$. This follows from the fact that the long-period terms of the order $m/es$ in the first approximation to $e$, $\varpi$ give rise to short-period terms when substituted in $u$, $t$, and that when we proceed to a second approximation, such terms can only produce a long-period term on multiplication by a short-period term containing the factor $e$. Thus, except in $\int dt/n$, such terms will not produce terms of orders greater than $m^2/s^2$ or $m^2/es$.

14. Next, we can have long-period terms formed from the products of long-period terms in the second derivatives of R by long-period terms in the elements.

Consider the first order long-period terms in the elements, *omitting those in* $t_0$ : they have the divisor $s$. When they are substituted in the equations which give the second-order terms, and when these equations are integrated, the results will contain the divisor $s^2$; the integration of the equation for $t_0$ will alone produce a divisor $s^3$. But the last equation depends on $\partial R/\partial t = n' \partial R/\partial f_1$, and the second derivatives are therefore of the form $\partial^2 R/\partial f_1 \partial \alpha$, where $\alpha$ is one of the elements. Now the derivatives $\partial R/\partial \alpha$ have already been obtained in terms of the angles $f$, $f_1$ in the first approximation and therefore the second derivatives $\partial^2 R/\partial f_1 \partial \alpha$ are at once found. Thus the terms with the divisor $s^3$ require very little additional calculation.

Next consider the case omitted in the last paragraph, namely, the long-period variations of $t_0$. When these are substituted in $\partial R/\partial \alpha$,

$\partial R/\partial t$, they require the calculation of $\partial^2 R/\partial a \partial t$, $\partial^2 R/\partial t^2$, that is, of $\partial^2 R/\partial a \partial f_1$, $\partial^2 R/\partial f_1^2$, which, as remarked in the previous paragraph, are at once obtained. These give rise to the divisors $s^3$, $s^4$, since the integration of the equation for $\delta a$ gives rise to a divisor $s$ and $\delta t_1$ depends on $D^{-1}\delta a$, and thus gives rise to the divisor $s^2$.

The secular variations of $e$, $\varpi$, $\Gamma$, $\theta$ combined with long-period terms in the second derivative of R will at most produce terms with the factors $m^2 v/s$ or $m^2/s^2$ in all the elements but $t_0$. The terms with factors $m^2 v/s^2$, or $m^2/s^3$ arise only from terms in $\delta t$ substituted in the equations for these elements, and from terms in the equation for $t_0$. They therefore only require calculations similar to those mentioned in the two preceding paragraphs.

Hence, *to obtain the long-period terms with factors* $m^2 v/s^2$, $m^2/s^3$, $m^2/s^4$ *in the second approximation we do not need to expand any second-order derivatives of* R: *they require only that the first-order derivatives be differentiated with respect to* $f_1$.

Further, if the disturbing planet contains a long-period perturbation, its chief portion will be present in $t$ only and this portion may evidently be added to the portion in $\delta t$ already considered.

15. The terms with the divisor of *order* $s^4$ can be obtained directly, as follows. They arise only from

$$D\delta\left(\frac{1}{2a} + R\right) = n'\delta\left(\frac{\partial R}{\partial f_1} Dt\right),$$

or, considering what has been said above, from

$$D\delta_2\left(\frac{1}{2a}\right) = n'^2 \frac{\partial^2 R}{\partial f_1^2} Dt \cdot \delta t_0, \quad . \quad . \quad . \quad (15.1)$$

in which $\delta t_0$ is a long-period term obtained in the first approximation from

$$D\delta_1\left(\frac{1}{2a}\right) = n' \frac{\partial R}{\partial f_1} Dt, \qquad D\delta t_0 = \delta_1\left(\frac{a^2}{\mu^2}\right),$$

or from

$$D\delta_1 a = -2n' \frac{\partial R}{\partial f_1} Dt, \qquad D\delta t_0 = \tfrac{3}{2}\delta a,$$

in which we put $1 = \mu = a^3 n^2$ and choose the mean value of $n$ as our unit, for the sake of brevity.

Let a long-period term in R be denoted by

$$R = A \cos(sv + s_1) \equiv A \cos(if + jf_1 + \alpha),$$

so that, for this term

$$\delta t_0 = -\frac{3An'j}{s^2} \sin(sv + s_1) \equiv B \sin(sv + s_1). \quad . \quad (15.2)$$

Let another (or the same) long-period term in R be denoted by

accented letters. Then the second approximation arising from the combination of the terms is, by (15.1),

$$-\mathrm{D}(\delta_2 a) = 2n'^2 j^2 \mathrm{A} \cos(s\mathrm{v} + s_1) \cdot \frac{3\mathrm{A}'n'j'}{s'^2} \sin(s'\mathrm{v} + s_1') + 2n'^2 j'^2 \mathrm{A}' \cos(s'\mathrm{v} + s_1') \cdot \frac{3\mathrm{A}n'j}{s^2} \sin(s\mathrm{v} + s_1),$$

giving the two terms,

$$\delta_2 t_0 = +\frac{9}{2} n'^3 \frac{\mathrm{A}\mathrm{A}'jj'}{(s \pm s')^2}\left(\pm\frac{j}{s'^2} + \frac{j'}{s^2}\right) \sin\{(s\mathrm{v} + s_1) \pm (s'\mathrm{v} + s_1')\}.$$

If $s\mathrm{v} + s_1 = s'\mathrm{v} + s_1'$, $\mathrm{A} = \mathrm{A}'$, we obtain

$$\delta_2 t_0 = +\frac{9}{8} \frac{n'^3 j^3 \mathrm{A}^2}{s^4} \sin(2s\mathrm{v} + 2s_1).$$

On account of (15.2), these respectively become

$$\begin{aligned} \delta_2 t_0 &= +\frac{1}{2} \frac{\mathrm{B}\mathrm{B}'n'}{(s \pm s')^2}(\pm js^2 + j's'^2) \sin\{(s\mathrm{v} + s_1) \pm s'\mathrm{v} + s_1')\}, \\ \delta_2 t_0 &= +\frac{j\mathrm{B}^2}{8} n' \sin(2s\mathrm{v} + 2s_1). \end{aligned} \qquad (15.3)$$

If $n'\mathrm{B}_1' \sin(s'\mathrm{v} + s_1')$ be a long-period perturbation of the disturbing planet additive to $v'$ and therefore to $f_1$, (15.1) becomes

$$\mathrm{D}\delta_2\left(\frac{1}{2a}\right) = n'^2 \frac{\partial^2 \mathrm{R}}{\partial f_1^2} \mathrm{D}t \cdot \mathrm{B}_1' \sin(s'\mathrm{v} + s_1').$$

Proceeding as above, we obtain the following results corresponding to (15.3) :—

$$\begin{aligned} \delta_2 t_0 &= +\frac{1}{2} \frac{\mathrm{B}_1\mathrm{B}_1'n'}{(s \pm s')^2} js^2 \sin\{(s'\mathrm{v} + s_1') \pm (s\mathrm{v} + s_1)\}, \\ \delta_2 t_0 &= +\frac{j\mathrm{B}\mathrm{B}_1'}{8} n' \sin(2s\mathrm{v} + 2s_1). \end{aligned} \qquad (15.4)$$

It is understood that in the formulæ of this paragraph the portions depending on the divisors $s$, $s^2$, $s^3$ are neglected, only those with divisors $s^4$ being retained.

16. As an example of the use of the formulæ of § 15, I shall find the second approximation to the coefficients of the terms depending on the argument $10g' - 4g$ in the theory of the mutual perturbations of Jupiter and Saturn, from the coefficients of the terms depending on the argument $5g' - 2g$. Since the coefficient of a term of very long period is nearly the same whether $t$ or v be used as the independent variable, we can make use of Hill's numerical results.*

* *Amer. Eph. Papers*, vol. iv. (1890).

For the terms with argument $5g' - 2g$, Hill gives :—*

For Jupiter, $n\delta z = + 1196''$ sin A ; for Saturn, $n'\delta z = - 2906''$ sin A,

where $A = 5g' - 2g - 82''t + 68°8'$. In applying the formulæ it is to be noticed that $n\delta t_0 = - n\delta z$, since the perturbations are here added to v instead of to $nt$. Hence $B = - 1196''$. On the other hand, $n'B_1'$ is defined as the coefficient of the term additive to $n't$ in $v'$, so that $B_1' = - 2906''/n'$. Whence, since $j = 5, n' \doteq \frac{2}{5}n \doteq \frac{2}{5}$ with the chosen units, we obtain for the sum of the two portions given by the second of the formulæ (15.3), (15.4), for Jupiter,

$$\delta_2 t = 5 \cdot \tfrac{2}{5} \cdot \tfrac{1}{8} \cdot \tfrac{11}{2}\tfrac{9}{0}\tfrac{6}{6}\tfrac{}{2}\tfrac{}{6}\tfrac{}{5} (1196'' + \tfrac{5}{2} \cdot 2906'') \sin 2A = 12''{\cdot}3 \sin 2A.$$

In a similar manner, for Saturn we get

$$\delta_2 t = - 2 \cdot \tfrac{5}{2} \cdot \tfrac{1}{8} \cdot \tfrac{2}{2}\tfrac{9}{0}\tfrac{0}{6}\tfrac{6}{}\tfrac{}{2}\tfrac{}{6}\tfrac{}{5} (2906'' + \tfrac{2}{5}\, 1196'') \sin 2A = - 30''{\cdot}9 \sin 2A.$$

But for the first and last approximations to the terms with argument 2A, Hill finds coefficients : †

For Jupiter, $+ 1''{\cdot}4, - 11''{\cdot}0$; for Saturn, $- 3''{\cdot}4, + 26''{\cdot}8$,

so that the approximations after the first yield the additional parts $- 12''{\cdot}4, + 30''{\cdot}2$, in the respective cases. When the signs of these are changed to accord with the notation of this paper, we see how closely the above approximate calculation gives these coefficients. In general, the longer the period, the better is the approximation.

### *Transformation to the Time as Independent Variable.*

17. For most purposes this transformation is unnecessary since a place or a set of tables can be calculated with v as the independent variable as easily as with $t$. If, however, it be needed, the following outline will show the method of procedure.

Suppose that we have found

$$t = \frac{1}{n_0}(v - \epsilon - E_v - P_v),$$

or, with $n_0 t + \epsilon \equiv w$,

$$w = v - E_v - P_v, \quad . \quad . \quad . \quad . \quad (17.1)$$

in which $E_v$ is the equation of the centre in terms of the true anomaly $v - \varpi$ and $P_v$ constitutes the sum of the perturbations.

We may regard (17.1) as an elliptic equation with "mean longitude" $w + P_v$, so that its reversion, by well-known formulæ, gives

$$v = w + P_v + E_{w+P-\varpi}, \quad . \quad . \quad . \quad (17.2)$$

where $E_{w+P-\varpi}$ is the equation of the centre with "mean anomaly"

* *Loc. cit.*, pp. 560, 564. The device used by Hill on p. 404 in order to obtain the best approximate motion of the argument should be adopted.

† *Loc. cit.*, pp. 105, 107, 561, 569.

$w + P_v - \varpi$. In this formula $P_v$ contains the disturbing mass as factor, and we can therefore substitute in it the elliptic value of $v$, namely,

$$v = w + E_{w-\varpi}.$$

This substitution constitutes the main part of the numerical work, but in asteroid calculations the number of terms to be transformed is usually not very great, and for most of them it is sufficient to put $v = w + 2e \sin(w - \varpi)$. There may be a second approximation necessary for a few terms: this, however, presents no difficulties.

The analogy to Hansen's form mentioned in § 1 is made evident by the transformation. Hansen adds the perturbations to the mean anomaly: substantially the same thing is done in this transformation, but these perturbations, instead of being calculated in terms of the time, are calculated in terms of v.

*Yale University:*
1929 *May.*

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## *The Masses, Luminosities, and Effective Temperatures of the Stars.*

By E. A. Milne.

1. In this paper I investigate the relations between the masses, luminosities, and effective temperatures of the stars from a standpoint which is philosophically different from that adopted by Professor Eddington in his well-known writings. The main conclusions are that it is not possible to infer from the observed masses, luminosities, and temperatures that the interiors of stars are necessarily composed of perfect gas; and that it is not possible to deduce the value of the absorption coefficient for the stellar interior. Instead, we are led to infer a single definite fact concerning the internal density-distribution.

### I. *Introduction.*

2. Contemporary physics makes progress by discarding ideas corresponding to quantities which experience shows can never be observed. The "velocity of a system through the aether" proves to be an "unobservable"; consequently a dynamics must be constructed which avoids this concept, and the resulting theory of frames of reference is relativity. Einstein's principle of relativity takes the place of the "unobservable" velocity through the aether. The simultaneous pair (position, velocity) when attached to an electron proves to be an unobservable; quantum mechanics therefore discards this concept, and Heisenberg's "uncertainty principle" takes its place. It may be necessary to introduce into our equations symbols corresponding to unobservables, but the physical content of the assertion which results from a piece of mathematico-physical reasoning must be a relation between observables only. It frequently appears, when this mode of

thought is followed, that the same results are deduced as before, but they are now shown to be true independently of the introduction of concepts corresponding to unobservables. Theory is enriched by being pruned of unnecessary assumptions.

The time seems ripe for applying this method to the structure of stars. The interior of a star can *never* be directly observed. It is not of course an unobservable in the same way as the velocity through the aether or the simultaneous (position-velocity) pair of an electron, for we do not doubt its " real existence " in the sense customary in physics. Yet the same general march of ideas impels us to inquire what are the relations between the " observables " of a star which are independent of arbitrary assumptions about the unobservable interior.

3. The observables of a star are its mass M, its luminosity L, and its effective temperature $T_1$; L and $T_1$ are equivalent to a determination of its radius $r_1$ by the relation $L = 4\pi r_1^2 \sigma T_1^4$, where $\sigma$ is Stefan's constant, for the definition of the effective temperature $T_1$ is that $\sigma T_1^4$ is the energy leaving each square centimetre of the star's surface. Actually the effective temperature has been determined for very few stars,* but we believe that the colour-temperatures which have been determined for many stars give us a fair knowledge of the effective temperatures.

We know further (1) that the star is in mechanical equilibrium, (2) that its outer layers are gaseous, (3) that its outer layers are in radiative equilibrium. For all these we have observational support: (1) follows because the stars hold together; (2) follows from their observed spectra; (3) is in fact only known from direct observation in the case of the sun (from the observed variation of brightness over the disc), but it seems legitimate to assume it for other stars. We shall find that (1), (2), and (3) are sufficient to yield a relation between L, M, $T_1$ and the internal structure; a specific theory of the interior is not required.

We have also a fair working knowledge of the physics of stellar atmospheres. The fact that the observed spectra of the stars fall into a linear sequence confirms both qualitatively and quantitatively Saha's theory of high-temperature ionization. The quantitative relations—the observed temperatures of maxima—amount in principle to a fixation of the photospheric opacity † of the star, though the values of the opacity so far determined may be liable to considerable uncertainty.

In addition to the foregoing, when we consider the stars as a whole we have the two sets of facts which are summed up in the observed mass-luminosity law and the observed luminosity-colour law. The former of these was discovered by Eddington, but it must be remembered that it has never received theoretical explanation; the formula derived by Eddington allows a star of given M to have an arbitrary luminosity by appropriate choice of its effective temperature $T_1$, and it has never been explained theoretically why the effective temperatures of the stars have their observed order of magnitude. How exact is the correlation between mass and luminosity for the majority of the stars remains for

* For the sun, by direct observation, and for a few other stars by measurement of their angular diameters.

† See Bakerian Lecture (1929), *Phil. Trans. Roy. Soc.*, A. **228**, 421.

future investigation, but the existence of a fair degree of correlation is a fact we must take into account. The luminosity-colour relation, summed up in the Hertzsprung-Russell diagram and exhibited more particularly (for main sequence stars) by Hertzsprung's recent studies * of the Pleiades, is very well established.

The general problems are (*a*) whether any of these relations can be explained or predicted in the light of present-day physics; (*b*) what information they yield as to the state of affairs in the interior of a star.

4. It is customary to speak of "luminosity-formulæ," and we begin our inquiry by asking what is meant by "calculating" the luminosity of a star. There are in principle two ways of calculating the luminosity of a star.

I. We may reconstruct as best we can the physical conditions—temperature, density, state of ionization, etc.—holding in the interior and then attempt to calculate what energy will be liberated under these conditions. It is known from various lines of evidence—astronomical, geological, and biological—that gravitational contraction is insufficient, and we deduce that the energy liberated must come from subatomic sources. If we could calculate the rate of liberation of subatomic energy per gram of matter under stellar conditions, simple addition through the whole mass of the star would give its luminosity in the steady state. Unfortunately, in the present state of physics, we know too little of the laws governing the rate of liberation of subatomic energy to be able to perform the calculation. We are therefore compelled to use another method.

II. We can build up a star in mechanical equilibrium of the same mass as the given star. We then let it *cool freely* for a short period *dt*. The different parts of the star interchange energy, and the star as a whole radiates to space. In the process some parts lose energy and cool down; others may gain energy and warm up. We now introduce a distribution of sources (and if necessary sinks) of energy through the star just sufficient to restore the star to its state at the beginning of the interval *dt*. The algebraic sum of the sources and sinks is equal to the amount radiated by the star as a whole to space. We then assume that the actual star is provided with just this distribution of sources and sinks. Its luminosity is then determined. The result of the calculation will be a formula connecting L with the mass M and any other quantities we find it necessary to introduce to build up the star in equilibrium and calculate its cooling. The important point to notice is that the problem is essentially a cooling problem. In any cooling problem boundary conditions are paramount, and they prove to be so in stellar problems. We must not treat the problem as a cooling problem for the innermost $\frac{9}{10}$ of the radius, and then assume the energy flowing past this point can flow through the outer $\frac{1}{10}$ of the radius without affecting the conditions inside.† That would be to change horses in mid-stream. We must carry our cooling problem to the bitter end.

* *M.N.*, **89**, 674, 1929.

† For an illustration taken from physics, see Appendix.

5. Method II. is in fact the method always followed in calculations of stellar luminosity, though it is not always analysed in this way. The analysis removes a difficulty sometimes felt respecting luminosity-formulæ: how is it, since we know nothing of subatomic energy, that we can make a calculation of the luminosity of a star? The answer is that we cannot. All we can do is to obtain a relation between L, M and other characteristics of a star. There are an infinite number of ways of building up a star in equilibrium, and in each the star cools at its own rate. All we can do is to link up L and M with the architecture of the star. We cannot hope by an analysis of a cooling problem to explain anything so fundamental as the mass-luminosity law. This must depend on deep-seated processes which are entirely beyond any discussion of equilibrium states. When we are discussing equilibrium states we must regard L and M as independent variables; a certain aggregate M generates a certain energy L per second—what is the physical state of the aggregate when in equilibrium? We might expect that the question would be answered by our being able to find the effective temperature $T_1$ of the spherical aggregate of matter. Analysis proves this is not so; $T_1$ is not fixed when we are given simply L and M—the problem still has a degree of flexibility.

That L and M are independent variables as far as equilibrium considerations go has been frequently pointed out by Sir James Jeans.* This is borne out by Professor Eddington's final luminosity-formula, according to which a star of given mass M can have any luminosity by adjustment of its effective temperature. I prefer to put the result in the form that a star of given luminosity L can have any mass M greater than a certain minimum.† This is easily proved as follows. Take any model of a star for which the luminosity L for a certain mass M has been calculated by treating the problem as a cooling problem. For example, take Eddington's particular solution for the star with a point-source at its centre. Insert the required sources,‡ and endow them with the property of emitting independently of physical conditions; they may be pictured as radioactive sources. Now bring up from outside an arbitrary mass M′ and scatter it gently gram by gram over the surface of the star, the mass M′ being "dead matter" that cannot generate energy. The new mass M+M′ must settle down into a new configuration of equilibrium—there is nothing else for it to do. The weight of the added mass will compress the pre-existing star; radiation-pressure (derived from L) insufficient to blow up M will certainly be insufficient to blow up M+M′, and gravity will bind the mass together. Thus starting with (L, M) we can always construct (L, M+M′); the only questions are: What is the effective temperature of the new system and how do the old sources of energy distribute themselves through the new star? In the case of the star with the point-source of energy the distribution is known, and only the first

* E.g. *Astronomy and Cosmogony* (1928), p. 83.

† A sufficiently small mass might be blown to infinity by a given luminosity under certain circumstances.

‡ I assume no sinks are required. They do not occur in Eddington's models.

of these two questions presents itself. Professor Eddington's solution of this problem is only one special solution ; there are an infinity of other solutions of the same luminosity and arbitrarily greater mass, constructed by the above process. It is clear that we can infer nothing as to the direct connection of mass with luminosity by considering point-source models.*

6. This process is applicable to any model with a finite radius. For example, we can scatter " dead " matter not too opaque over the surface of Eddington's model " $\kappa\eta$=constant." † The new star will of course in general not satisfy this relation ; convection currents may originate in the outer layers and diffuse the energy-sources through the added " dead " material until a new state of radiative equilibrium (without convection currents) is attained.

Consideration of the model "$\kappa\eta$=constant " gives us further insight into the equilibrium of a star. Let us suppose that $\kappa\eta$ is rigorously constant from centre to photosphere. Then since $\eta$ is equal to unity at the photosphere the constant value of $\kappa\eta$ is equal to $\kappa_1$, the value of $\kappa$ at the photosphere. The peculiarity of this model ‡ is that the mechanical equilibrium is of the type known as " neutral " ; the star can exist in mechanical equilibrium with any radius and so any effective temperature. By Eddington's calculations, the star cools at the rate $4\pi c\mathrm{GM}(1-\beta)/\kappa_1$ ergs per second, where $1-\beta$, the ratio of radiation-pressure to total pressure through the star, is a constant depending simply on the mass and molecular weight. Now $\kappa_1$ will depend on the pressure and temperature at the photosphere, and so on the radius of the star. § The star therefore adjusts its radius until the above expression is equal to L, the rate of liberation of energy throughout the interior. It can make this adjustment quite easily since its mechanical equilibrium is neutral. If, for example, in any configuration of mechanical equilibrium the photospheric absorption coefficient is too small, the cooling (radiation to space) will exceed the supply of energy and the star will alter its radius until the photospheric absorption coefficient has the desired value given by $\mathrm{L}=4\pi c\mathrm{GM}(1-\beta)/\kappa_1$, L and M being given arbitrarily (subject to a possible upper limit for L). We see from this example that when the mechanical equilibrium is neutral the surface conditions can have a completely dominant effect.

If $\kappa$ is independent of temperature and pressure, and $\epsilon$, the rate of

* Similarly, given a star (L, M) we can always construct another (L−L′, M) of smaller luminosity and equal mass by simply diminishing the rate of supply of energy from the sources. Initially each element of the star has acceleration zero, and the star will initially cool at the same rate as before ; this means that cooling will now exceed the energy-supply and the star will slowly contract. For example, in the point-source model, diminution of the strength of the point-source will result in a condensation of the star towards the centre.

† $\kappa$ is the mass-absorption coefficient ; $\eta$ is a parameter measuring the concentration of energy-liberation toward the centre and equal to unity at the boundary. $\eta$ is defined in § 18 below.

‡ We shall see that this peculiarity is shared by all models not possessing a singularity at the centre. See § 16 below.

§ From any physical formula for $\kappa$, $\kappa_1$ is readily calculated in terms of $\mathrm{T}_1$ and $1-\beta$. See § 20 below, equation (51).

liberation of energy per gram, is a constant, the equilibrium is more delicate still. At first sight it might appear that this star, according to Eddington's solution, could rest in equilibrium with any radius. This is true with Eddington's boundary conditions, but the equilibrium is again neutral and the configuration is very sensitive to the boundary conditions. In Eddington's solution the total pressure P has been taken to tend to zero at the boundary $\rho = 0$.* Actually if a star has a finite radius ($\rho = 0$ for finite $r$), the conditions of radiative equilibrium require that P shall tend to a small constant $P_0$ at the boundary. Professor Eddington has argued that this small constant, of the order of 1 dyne cm.$^{-2}$, can be neglected in comparison with the internal pressures of the order of $10^{12}$ dynes cm.$^{-2}$. But when the equilibrium is neutral a small change in boundary conditions may have a devastating effect. The value of $P_0$ is $\frac{1}{2} \cdot \frac{1}{3} a T_1^4$, whilst the boundary radius $r_0$ is given by $L = 4\pi r_0^2 \cdot \frac{1}{4} a c T_1^4 = 6\pi c r_0^2 P_0$. If the condition $P \to P_0 (P_0 \neq 0)$ is imposed, it is found that $\rho = 0$ only at infinity, thus contradicting our supposition that the star had a finite radius. We are then driven back on the solution with $P_0 = 0$, namely, Professor Eddington's solution, but we see that in order to satisfy the boundary conditions the radius must now be infinite. The state is now one of infinite diffusion. The relation between L, M, and $\kappa$ for this model is a particular case of Eddington's four-term relation between L, M, $\bar{\kappa}$, and $\bar{\eta}$, in which $\bar{\eta}$ has been put equal to unity.

If, on the other hand, "$\kappa\eta$" is not constant, the solution does not hold, and the equilibrium requires re-investigation. This is carried out below. We shall find that the photospheric absorption coefficient $\kappa_1$ is just as significant in the general case as in the model "$\kappa\eta$ = constant."

This digression serves to bring out two points of importance: ($\alpha$) that a star of given luminosity can in general have any mass above a certain limit; ($\beta$) that the boundary conditions must be taken into account in cases of neutral equilibrium.

7. When we proceed to calculate the cooling of any model whatever we find that using only assumptions (1), (2), and (3) above we are led to a four-term relation connecting L, M, the photospheric opacity $\kappa_1$ and a certain coefficient C. This coefficient C is a number depending on the complete march of the density-distribution from centre to photosphere. It depends on the general architecture of the star. Professor Eddington has in general investigated stars of given $\eta$, $\eta$ being a parameter measuring the distribution of energy-sources, but the trouble with this method is that we cannot derive a luminosity-relation in which all the terms are calculable. By beginning with the density-distribution, or rather a certain function $\phi$, which fixes the density-distribution, we are enabled to obtain our luminosity-relation explicitly.

The equilibrium for a given $\phi$ would be neutral if $\kappa_1$ were a constant; the star could rest in equilibrium with any radius.† But the

* $\rho$ denotes the density.

† Apart from more delicate considerations of the type mentioned in § 6.

mode of dependence of $\kappa_1$ on temperature and pressure in the photosphere anchors the star of given L, M, and $\phi$ to a particular radius, *i.e.* to a particular $T_1$. For given L, M, and $\phi$ the star must adjust its radius until $\kappa_1$ is just right.

8. Stars of given $\phi$ form a "homologous family," of a definite density-distribution, doubly infinite in number, corresponding to varying mass and radius. Any particular actual star will be a member of some one homologous family, namely, that for which the density-distribution is the same as that for the star under consideration. The question arises whether we can determine the homologous family to which the star belongs. From its observed L and M and a knowledge of its $\kappa_1$,† we can by means of our four-term relation calculate its C. This does not suffice to fix $\phi$, for $\phi$ is a function whilst C is a single number associated with $\phi$. The value of C is, however, a definite piece of information about $\phi$, equivalent to the mean gradient of $\phi$ from the centre to photosphere. When $\phi$ is constant, its value (from its definition) is unity, and the density-distribution $\phi = 1$ is that of Eddington's model, the Emden polytrope $n = 3$. Since the value of C fixes the departure of $\phi$ from constancy, it fixes the average deviation of the density-distribution from that of the polytrope $n = 3$. We could in this way determine the coefficient C from observation for every star.

I have found it convenient, however, to proceed differently. The photospheric opacity $\kappa_1$ is not accurately determined, so that the values of C deduced by the foregoing procedure would not be very trustworthy. What we are really interested in is the variation in type of density-distribution from star to star. In order to have a basis of comparison it is of interest to construct a complete homologous family of stars, *i.e.* tabulate their masses, luminosities, and temperatures, and then compare the actual stars with this family. The family I choose to construct is that family which contains two members of the same mass, luminosity, and effective temperature as the Sun and Capella, respectively. Call these two members Sun* and Capella*. These two hypothetical stars provide us with two equations, namely, our four-term relation between L, M, $\kappa_1$, and C for each star. The value of C is the same for each star, for they belong to the same homologous family. The values of $\kappa_1$ will be different, but they involve, according to the theory of the photospheric absorption coefficient, a single undetermined constant $\alpha$, a physical constant depending only on the properties of stellar material in photospheres. We have thus two equations and two unknowns, namely, C and $\alpha$. Solving for C and $\alpha$ we have the data to construct our homologous family of which two members, Sun* and Capella*, are indistinguishable observationally from the Sun and Capella. We then tabulate the luminosities, masses, and effective temperatures of all other members of the family.

9. This family of hypothetical stars may now be compared with the

† Actually C is involved in $\kappa_1$; to evaluate C for the star we use the observed L, M, and $T_1$, and a constant $\alpha$ defining the photospheric opacity from pure physics. The details will be clear later.

actual stars. I originally constructed the table merely as an experiment, as a standard of comparison. It then appeared that the table fairly accurately predicted the luminosities of all the known stars in the heavens ; † that is to say, for a star of given M and $T_1$, the luminosity of the corresponding star in our hypothetical family is closely the same as the luminosity of the actual star. In other words, the actual stars are observationally indistinguishable from the members of a certain homologous family.

This is equivalent to the discovery that the coefficient C has the same value for all the observed stars provided the coefficient $a$ defining the photospheric opacity is taken to have a certain value. The value of the photospheric opacity necessary to attribute to Sun* and Capella* is about twice that which I have independently deduced for Sun and Capella from spectroscopic considerations.‡ If the spectroscopic opacities of the photospheres of Sun and Capella and the photospheric opacities necessary to attribute to Sun* and Capella* had come out of entirely different orders of magnitude, our special homologous family would have been an entirely artificial construct ; the members of our homologous family would have had to be endowed with a photospheric opacity different from that of real stellar material. The fact of the agreement is, however, profoundly significant. It is clear that we could have reversed the procedure. Taking the opacity given by spectroscopic data, or something close to it, we could have computed the value of C, using our four-term relation, for each observed star from its observed mass, luminosity, and effective temperature ; the previously noted agreement between our table of luminosities and the observed luminosities now implies that the values of C would have come out approximately the same for all the stars.

This means that as far as the coefficient C discloses information about the function $\phi$, the stars (giants and main sequence stars) do in fact form a single homologous family, with a common density-distribution.§

10. This is the main positive result of the present paper. Its evolutionary implications are considered in a paper which is to follow. It is no part of the purpose of the present paper to discuss why the stars should turn out to have a common density-distribution. We recall, however, that the only hypotheses we have made are that the stars are in mechanical equilibrium, that their outer layers are composed of perfect gas, and that their outer layers are in radiative equilibrium. We have not assumed that their interiors are composed of perfect gas, still less do we need to know their mean molecular weight ; we have made no assumption about the distribution of energy-generation in their interiors. For all our analysis is concerned they may be liquid in their interiors, as supposed by Sir James Jeans. Clearly the observed relations between the masses, luminosities, and temperatures

† With the exception of the white dwarfs. See § 28.

‡ Bakerian Lecture, *Phil. Trans. Roy. Soc.*, 1929.

§ Not of course a common density. A common density-distribution is best envisaged as a common ratio of mean density to central density.

of the stars, now partly described in the form that they belong to a homologous family, cannot be taken to imply that the interiors are composed of perfect gas. It may well be that some stability criterion, of the type discussed by Jeans, comes in to compel the stars to take up a common density-distribution ; or it may be that there is some reaction of physical conditions on the distribution of energy-liberation —the distribution of sources and sinks necessary for the observed structure to be a steady-state one. These are problems beyond the scope of the present investigation.

11. It should be pointed out that the present investigation, in the first instance, explains nothing—neither the observed mass-luminosity law nor the Hertzsprung-Russell diagram. We simply discover a common feature of the stars not depending on hypotheses about their interiors but deduced from observation. But a succeeding paper will show that this common feature combined with the empirical mass-luminosity law goes some way towards accounting for the Hertzsprung-Russell diagram and other stellar characteristics.

It should also be pointed out that the results of the paper after a certain point depend on an assumed formula for the photospheric absorption coefficient. But the general principles of the paper are quite independent of any special theory of the photospheric absorption coefficient. Any other theory could be introduced without the slightest difficulty, though it might then be found that the stars did not form a homologous family. The main point is that the method of the present paper permits an actual determination of a characteristic connected with the architecture of the stars, from observation.

12. Professor Eddington's luminosity formula in its well-known form

$$L = \frac{4\pi cGM(1-\beta)}{\overline{\kappa\eta}},$$

where $1-\beta$ is determined in terms of M, is a three-term relation, connecting two " observables " L and M with an " unobservable " $\overline{\kappa\eta}$ ; thus it determines $\overline{\kappa\eta}$ from an observed L and M. But it affords no separation between $\kappa$ and $\eta$ as it stands. In view of what has been shown earlier about the independence of L and M as variables this formula cannot be held to determine the luminosity of a star ; also $\eta$ is unknown, so the formula is not competent to isolate the value $\bar{\kappa}$, the opacity of the interior. In Eddington's formula $\overline{\kappa\eta}$ is an undefined mean. Formula (32) below is a similar formula connecting $1-\beta_c$, the central value of $1-\beta$, with a determined mean value of $\kappa\eta$ ; on the hypothesis of an interior composed of perfect gas it is possible to estimate roughly $1-\beta_c$ from our knowledge of the function $\phi$. If now the interior opacity is calculable by pure physics, formula (32) affords a value of $\bar{\eta}$ for any actual star. It tells us in fact something about the distribution of sources necessary to maintain the star in a steady state. This appears to me to be the true significance of Eddington's " luminosity " formula or its more precise form (32) ; either

determines the source-distribution from the observed L and M and the computed interior opacity.

We proceed to give the mathematical analysis.

## II. *General Theory of Stellar Equilibrium.*

13. *Mechanical Equilibrium.*—Let P be the total pressure (gas-pressure + radiation-pressure) at a point in a non-rotating star distant $r$ from the centre ; $\rho$ the density at this point ; $M(r)$ the mass enclosed by a sphere of radius $r$ ; G the constant of gravitation. Then the equation of mechanical equilibrium is

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho, \quad . \quad . \quad . \quad . \quad (1)$$

where

$$M(r) = \int_0^r 4\pi r^2 \rho dr \quad . \quad . \quad . \quad . \quad (2)$$

Introducing (2) in (1) and differentiating we have

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \quad . \quad . \quad . \quad (3)$$

Let $P_c$, $\rho_c$ be the central values of P and $\rho$. Put

$$P = P_c y, \qquad \rho = \rho_c z \quad . \quad . \quad . \quad (4)$$

so that $y = 1$ and $z = 1$ at $r = 0$. Then

$$\frac{P_c}{4\pi G{\rho_c}^2}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{z}\frac{dy}{dr}\right) = -z \quad . \quad . \quad . \quad (5)$$

We remove the central constants by making the substitution

$$r = \left(\frac{P_c}{4\pi G{\rho_c}^2}\right)^{\frac{1}{2}} x \quad . \quad . \quad . \quad . \quad (6)$$

Then (5) becomes

$$\frac{1}{x^2}\frac{d}{dx}\left(\frac{x^2}{z}\frac{dy}{dx}\right) = -z \, . \quad . \quad . \quad . \quad (7)$$

This equation is not sufficient to fix the density-distribution in the star ; we have two unknowns, $y$ and $z$, and only one equation. In any actual configuration of equilibrium $y$ and $z$ will vary smoothly from centre to boundary, and if they are plotted against one another a smooth curve will result. Let

$$y = f(z) \quad . \quad . \quad . \quad . \quad (8)$$

be the equation of this curve. Then the density function $z(x)$ is that solution of

$$\frac{1}{x^2}\frac{d}{dx}\left(\frac{x^2}{z}\frac{df(z)}{dx}\right) = -z \quad . \quad . \quad . \quad (9)$$

which satisfies

$$z = 1, \qquad \frac{dz}{dx} = 0 \quad \text{at} \quad x = 0.$$

We note that $f(1) = 1$ by definition.

14. *Homologous Families.*—To every arbitrary function $f(z)$ for which $f(1) = 1$ there corresponds a definite density-distribution $z(x)$, the solution of (9). All stars having this density-distribution we speak of as the "homologous family defined by $f$." The classical researches of Lane, Jeans, and others have investigated the properties of homologous *series* of stars, *i.e.* a series of configurations of equilibrium of constant mass. A "homologous family" is a generalisation of this concept, as it includes stars of all masses and all radii having the same density-distribution.

The function $z(x)$ may at first increase or decrease with $x$. For functions $f$ corresponding to physically existing stars $z(x)$ will ultimately decrease to zero for some finite value of $x$. Let $x_1$ be this value. Then the radius $r_1$ of the star is given by

$$r_1 = \left(\frac{P_c}{4\pi G\rho_c^2}\right)^{\frac{1}{2}} x_1 \quad . \quad . \quad . \quad . \quad (10)$$

We see that given $f$, *i.e.* given the homologous family, we still have two constants at our disposal, namely, $P_c$ and $\rho_c$. Thus there is a double infinity of stars of a given homologous family.

Let us now find the mass M of a given member ($P_c$, $\rho_c$) of a given homologous family. We have

$$M = 4\pi\int_0^{r_1} \rho r^2 dr = 4\pi\rho_c\left(\frac{P_c}{4\pi G\rho_c^2}\right)^{\frac{3}{2}}\int_0^{x_1} zx^2 dx,$$

or, using the differential equation (9),

$$M^2 = \frac{P_c^3}{4\pi G^3\rho_c^4}\left[\frac{x^2}{z}\frac{df(z)}{dx}\right]^2_{x=x_1} \quad . \quad . \quad . \quad (11)$$

where the suffix $x = x_1$ denotes that the expression is to be evaluated at $x = x_1$. Thus, for members of a given homologous family,

$$M \propto P_c^{\frac{3}{2}}/\rho_c^2 \quad . \quad . \quad . \quad . \quad (12)$$

Given the mass, one constant remains still at our disposal. By (10) this may be taken to be the radius.

As an example of a property of a homologous family we will find the ratio of mean density $\bar{\rho}$ to central density $\rho_c$. We have

$$\frac{\bar{\rho}}{\rho_c} = \frac{M}{\frac{4}{3}\pi r_1^3\rho_c} = \frac{4\pi\int(\rho/\rho_c)r^2dr}{\frac{4}{3}\pi r_1^3} = \frac{3\int zx^2dx}{x_1^3} = -\frac{3}{x_1}\left[\frac{1}{z}\frac{df(z)}{dx}\right]_{x=x_1} \quad (13)$$

Thus $\bar{\rho}/\rho_c$ is the same for all members of a homologous family. Again, from (10) and (12)

$$P_c \propto M^2/r_1^4 \quad . \quad . \quad . \quad . \quad (14)$$

15. So far we have only used the equation of mechanical equilibrium ; the star may have any equation of state, gaseous or liquid. *Let us now assume the outer layers are gaseous,* in accordance with observation. We proceed to calculate the ratio of radiation-pressure to total pressure in the outer layers.

Let T be the temperature at any point in the outer layers, $\mu$ the mean molecular weight at that point. Let $1-\beta$ be the ratio of radiation-pressure to total pressure at this point. (We do not assume $1-\beta$ is constant.) Then

$$(1-\beta)\mathrm{P} = \tfrac{1}{3}a\mathrm{T}^4 \quad . \quad . \quad . \quad . \quad (15)$$

and

$$\beta\mathrm{P} = (\mathrm{R}/\mu)\rho\mathrm{T} \quad . \quad . \quad . \quad . \quad (16)$$

where $a$ is the radiation-constant, R the gas-constant. Eliminating T we have

$$\frac{1-\beta}{\frac{1}{3}a\beta^4}\cdot\left(\frac{\mathrm{R}}{\mu}\right)^4 = \frac{\mathrm{P}^3}{\rho^4} = \frac{\mathrm{P}_c^{\,3}}{\rho_c^{\,4}}\frac{[f(z)]^3}{z^4} \quad . \quad . \quad (17)$$

It is known from the researches of Jeans * and Woltjer † that as we enter the star from outside, $1-\beta$ changes rapidly from its boundary value unity ‡ to a very slowly changing value which it reaches at the photospheric level ; thereafter $1-\beta$ changes very slowly as we go further inwards. For a reason which will appear later we are anxious to calculate the value of $1-\beta$ at the photosphere, which we will call $1-\beta_1$. At this level the left-hand side of (17) is finite whilst $z$ is very small ; hence $f(z)$ is very small. To avoid awkward limits we now introduce a parametric representation for $f(z)$, namely,

$$z = \theta^3 \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$f(z) = \phi(\theta)\theta^4 \quad . \quad . \quad . \quad . \quad (19)$$

The boundary value of $\theta$ is zero. We shall impose the restriction that $\phi(\theta)$ tends to a definite finite limit $\phi_1$ as $\theta \to 0$, and we shall henceforward consider our homologous families to be defined by the function $\phi(\theta)$. With this restriction on $\phi$ the solution of the differential equation no longer represents the actual density-distribution outside the photosphere, but $\phi$ may be chosen so that it does represent it from centre to photosphere. The value $\phi_1$ is practically attained at the photosphere. We shall later have to examine the effect of the layers outside the photosphere. This introduction of a restriction on $\phi$ is only a mathematical device, and it could be avoided by choosing a $\phi$ which, after becoming nearly constant for $\theta$ very small and decreasing, suddenly increased to infinity. We save algebra by treating the layers outside the photosphere separately.

* Jeans, *M.N.*, **85**, 199, 1925.

† Woltjer, *B.A.N.*, **2**, 171 (No. 66), 1924. See also Milne, *M.N.*, **85**, 784, 1925 ; and Bakerian Lecture, 1929, *Phil. Trans. Roy. Soc.*

‡ Gas-pressure is zero at the boundary, so the radiation-pressure is equal to the total pressure.

The differential equation (9) now becomes

$$\frac{4}{x^2}\frac{d}{dx}\left[x^2\frac{d\theta}{dx}\{\phi(\theta)+\tfrac{1}{4}\theta\phi'(\theta)\}\right]=-\theta^3 \quad . \quad . \quad (20)$$

the mass equation (11) becomes

$$M^2=\frac{P_c{}^3}{4\pi G^3\rho_c{}^4}\left[\frac{x^2}{\theta^3}\frac{d}{dx}\{\theta^4\phi(\theta)\}\right]^2_{x=x_1} \quad . \quad . \quad . \quad (21)$$

Neglecting the exceedingly small difference between the true mass inside the photosphere and the mass in our $\phi$-model, we may let $\theta\to 0$ in the factor on the right-hand side of (21), obtaining

$$M^2=\frac{4P_c{}^3}{\pi G^3\rho_c{}^4}x_1{}^4\phi_1{}^2\left(\frac{d\theta}{dx}\right)^2_1 \quad . \quad . \quad . \quad (22)$$

Lastly, applying (17) to the photosphere, we have

$$\frac{1-\beta_1}{\frac{1}{3}a\beta_1{}^4}\left(\frac{R}{\mu_1}\right)^4=\frac{P_c{}^3}{\rho_c{}^4}\phi_1{}^3 \quad . \quad . \quad . \quad (23)$$

We now note the remarkable fact that $P_c$ and $\rho_c$ are involved in the same way in (22) and (23), namely, in the form $P_c{}^3/\rho_c{}^4$. This is a mathematical coincidence, for (22) depends only on hydrostatic equilibrium whilst (23) depends on the peculiar forms of the gas- and radiation-laws. Eliminating $P_c{}^3/\rho_c{}^4$ between (22) and (23), we have

$$\frac{1-\beta_1}{\beta_1{}^4}\left(\frac{R}{\mu_1}\right)^4=\frac{\pi aG^3M^2}{12}\frac{\phi_1}{x_1{}^4(d\theta/dx)_1{}^2} \quad . \quad . \quad (24)$$

Since $\theta(x)$ is the solution of a differential equation containing $\phi$, $(d\theta/dx)_1$ depends on the complete march of $\phi$ from centre to photosphere, and hence so does $1-\beta_1$. Thus $1-\beta_1$ depends on the complete density-distribution, and not merely on the local value of $\phi$. We write

$$C=\frac{\phi_1}{x_1{}^4(d\theta/dx)_1{}^2} \quad . \quad . \quad . \quad . \quad (25)$$

16. Let us now introduce the assumption, supported by observation in the case of the Sun, that *the outer layers are in radiative equilibrium.* Let $L(r)$ be the net flux of energy across a sphere of radius $r$. Then the net flux per unit area, $L(r)/4\pi r^2$, is given by Eddington's equation

$$\frac{L(r)}{4\pi r^2}=-\frac{c}{\kappa\rho}\frac{d(\frac{1}{3}aT^4)}{dr} \quad . \quad . \quad . \quad (26)$$

where $\kappa$ is the mass-absorption coefficient. Combining this with the equation of mechanical equilibrium (1) and using (15) we have the relation, also due to Eddington,

$$L(r)=\frac{4\pi cGM(r)}{\kappa}\frac{d[(1-\beta)P]}{dP} \quad . \quad . \quad . \quad (27)$$

We wish to ascertain the flux of energy at the photosphere, *i.e.* the limit of $L(r)$ as $\theta$ tends to practically zero. Let this limit be L, the luminosity of the star. Then

$$L = \frac{4\pi cGM}{\kappa_1} \lim_{\theta \to 0} \frac{d[(1-\beta)P_c\phi(\theta)\theta^4]}{P_c d[\phi(\theta)\theta^4]}$$

$$= \frac{4\pi cGM(1-\beta_1)}{\kappa_1} \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Since $1-\beta_1$ is fixed by (24), we have now found the luminosity L for our model of given $\phi$. Equations (24) and (28) in combination form a four-term relation between L, M, $\kappa_1$, C which must be satisfied for any star.

Let us suppose we have some physical formula for $\kappa$ (the absorption coefficient in the photospheric layers); then $\kappa_1$, the value of $\kappa$ at the photosphere itself (where $T = T_1$), may be calculated by analysis of the equilibrium of the layers outside the photosphere, for given $g$ (gravity) and $T_1$, *i.e.* for given L, M, and $T_1$. It will in general be a function of $T_1$.* If $\kappa_1$ proved to be independent of $T_1$ (*e.g.* if it were a single physical constant), we could introduce it in (28), calculate $1-\beta_1$, and so from (24) determine C. Thus for a star of given L and M we could determine a single fact about the density-distribution, namely, that the expression on the right-hand side of (25) has a certain numerical value. But the radius of the star would be quite indeterminate. We have seen that owing to our having been able, by a mathematical coincidence, to eliminate $P_c$ and $\rho_c$ simultaneously, and so obtain $1-\beta_1$ as a function of M and C only, we should have still the actual value of $\rho_c$ at our disposal. The star would be in neutral equilibrium, on our model, and very close discussion of boundary conditions would be necessary to alter our model so as to give a determinate radius. But if $\kappa_1$ depends on $T_1$, the radius of the star is fixed. If $\phi$ is given, it must expand or contract (which it can do freely since its mechanical equilibrium is neutral) until the temperature $T_1$ at the photosphere is just sufficient to make $\kappa_1$ come right according to (28). Our problem is the inverse problem. From observed L, M, and $T_1$ we can compute $\kappa_1$ from any physical theory, derive $1-\beta_1$ from (28), and so from (24) compute C. This is the single piece of information we can derive about the density-structure on the hypotheses we have made—mechanical equilibrium of the star as a whole, gaseous outer layers in radiative equilibrium.

17. When $\phi(\theta) = \text{constant} = 1$, the fundamental differential equation (20) reduces to Emden's equation for the polytrope $n = 3$. C then has the same value as in Eddington's quartic equation. Thus the departure of C from the value characteristic of the polytrope $n = 3$ will be a measure of the departure of the density-distribution from that of Eddington's model. We can see in a general way that the value

* As well as of $g$. The effect of the pressure in the photosphere is duly taken account of below when we come to calculate $\kappa_1$.

deduced for C from observation is equivalent to the determination of the mean gradient of $\phi(\theta)$ between centre and photosphere. If $\phi(\theta)$ is not constant, the next simplest form it can take is the linear form

$$\phi(\theta) = 1 - \lambda(1 - \theta). \quad . \quad . \quad . \quad (29)$$

Inserting this in (20) we should have

$$\frac{4}{x^2}\frac{d}{dx}\left[x^2\frac{d\theta}{dx}\left\{1 - \lambda + \frac{5}{4}\lambda\theta\right\}\right] = -\theta^3 \quad . \quad . \quad (29')$$

We can imagine this solved for various values of $\lambda$, and then the value of $\lambda$ adjusted so that the value of C calculated from (25) comes out equal to that determined by observation. In this way we could acquire information about the general march of $\phi$ between centre and boundary. In the present paper, however, I make no attempt to determine the march of $\phi(\theta)$ from the observed C.

18. We will now digress to inquire how much more we can find out about the star if we assume it to be gaseous throughout and in radiative equilibrium throughout. If it is gaseous throughout, (17) holds throughout and we have

$$\frac{1 - \beta}{\frac{1}{3}a\beta^4}\left(\frac{R}{\mu}\right)^4 = \frac{P_c^3}{\rho_c^4}[\phi(\theta)]^3 \quad . \quad . \quad . \quad (30)$$

Now at the centre ($\theta = 1$), $\phi(\theta) = 1$. Applying (30) to the centre and substituting for $P_c^3/\rho_c^4$ from (22) we have

$$\frac{1 - \beta_c}{\beta_c^4}\left(\frac{R}{\mu_c}\right)^4 = \frac{\pi a G^3 M^2}{12}\frac{1}{\phi_1^2(d\theta/dx)_1^2} \quad . \quad . \quad (31)$$

Since, by the linear approximation for $\phi$ or otherwise, the march of $\phi(\theta)$ for the star under consideration could now be considered known, the value of $\phi_1$ could be estimated, and so the coefficient of $M^2$ on the right-hand side of (31) could be estimated. Hence $1 - \beta_c$ could be estimated.

If the star is in radiative equilibrium throughout, the flow of energy is given by (27). Solving (30) for $1 - \beta$ ($\phi(\theta)$ being now considered known from observation), we can insert for it in (27). Since $L(r) = \int_0^r 4\pi r^2 \epsilon dr$, the rate of liberation of energy at any point, $\epsilon$, could now be found by differentiation. In this way an estimate of the distribution of energy-liberation through the star is possible from observation. The method presupposes, however, an accurate knowledge of $\kappa$ at each point.

To bring this method into relation with Eddington's "luminosity-formula," integrate (27) from photosphere to centre. We find

$$(1 - \beta_c)P_c - (1 - \beta_1)P_1 = \int_{P_1}^{P_c}\frac{\kappa L(r)}{4\pi c G M(r)}dP.$$

This can be written approximately as

$$(1 - \beta_c) = \frac{1}{P_c}\int_0^{P_c}\frac{\kappa L(r)}{4\pi c G M(r)}dP.$$

Following Eddington by writing

$$L(r)/M(r) = \eta L/M,$$

we have

$$1 - \beta_c = \frac{L}{4\pi cGM} \cdot \frac{1}{P_c} \int_0^{P_c} \kappa\eta dP \quad . \quad . \quad . \quad (32)$$

We have seen how $1 - \beta_c$ may be estimated from (31). Hence (32) affords an estimate of $\kappa\eta$. Formula (32) is analogous to Eddington's "luminosity-formula" (§ 12), with the difference that the way of meaning-up $\kappa\eta$ is specified and that $1 - \beta_c$ is given by a certain quartic equation, namely (31), in which the coefficient of $M^2$, instead of having a hard-and-fast value, has a value estimated from observation.

If we have a physical theory of the absorption coefficient $\kappa$ in the interior, $\bar{\kappa}$ can be estimated, and so (32) evaluates $\bar{\eta}$. This seems to me to be the real significance of Professor Eddington's "luminosity formula," or its analogue (32); it determines $\bar{\eta}$ from an observed L and M if $\bar{\kappa}$ is known from pure physics, *i.e.* it determines the concentration of energy-liberation towards the centre from the observed data for an actual star, *granted a knowledge of the absorption coefficient.*

It should be observed that the method indicated in §§ 17, 18 depends on much more drastic hypotheses than does the determination of C alone, and it depends further on assuming some kind of formula for $\phi$ (such as the linear one, or $e^{\lambda(1-\theta)}$, or any simple expression containing an adjustable parameter). The important thing is the principle involved; L, M, and $\kappa_1$ provide definite but scanty information about $\phi$, and this combined with the assumption of a gaseous interior in radiative equilibrium provides some but still scantier information about $\eta$, if $\bar{\kappa}$ is known.*

19. We now turn back from speculations about the interior and continue our investigation where we left it at the end of § 16. We have to show how to calculate $\kappa_1$. Incidentally we shall derive a slight correction to (28) due to the effect of the layers outside the photosphere.

20. *The Mechanical and Radiative Equilibrium of the Photospheric Layers.*—Let $p$ be the gas-pressure, $p'$ the radiation-pressure at a point $r$ in the photospheric layers. Let F be the net flux of energy per unit area, $g$ the acceleration due to gravity; F and $g$ may here be taken to be independent of $r$. Then we have the equation of hydrostatic equilibrium

$$dp' + dp = -g\rho dr \quad . \quad . \quad . \quad . \quad (33)$$

and the equation of radiative equilibrium †

$$dp' = -\kappa\rho F/c \quad . \quad . \quad . \quad . \quad (34)$$

* The method of § 18 applies whatever equation of state is assumed for the interior. Given any equation of state for the interior, and any theory of the absorption coefficient $\kappa$ in the interior, we can always ascertain the distribution of sources in the interior for a $\phi$ compatible with the boundary conditions, *i.e.* for a density-distribution consistent with the observed L, M, and $T_1$.

† This equation is known to hold very closely right up to the boundary.

Hence

$$1 + \frac{dp}{dp'} = \frac{cg}{\kappa F} \quad . \quad . \quad . \quad . \quad (35)$$

Now assume the opacity of the photospheric layers is due to the photo-electric effect. Then *

$$\kappa = \frac{\alpha P_e}{T^{9/2}} . \quad . \quad . \quad . \quad . \quad (36)$$

where $\alpha$ is a physical constant and $P_e$ is the electron-pressure. Let $\bar{x}$ be the average number of electrons per ion ; † we assume it not to vary with depth in the layers now under discussion. Then

$$\frac{P_e}{p} = \frac{\bar{x}}{1 + \bar{x}} \quad . \quad . \quad . \quad . \quad (37)$$

Using $p' = \frac{1}{3}aT^4$ we have

$$\kappa = \frac{\alpha\bar{x}}{1 + \bar{x}} \left(\frac{\frac{1}{3}a}{p'}\right)^{9/8} p \quad . \quad . \quad . \quad (38)$$

Hence by (35)

$$1 + \frac{dp}{dp'} = \frac{cg}{F\alpha} \frac{1 + \bar{x}}{\bar{x}} \frac{1}{(\frac{1}{3}a)^{9/8}} \frac{p'^{9/8}}{p} . \quad . \quad . \quad (39)$$

To solve this make the substitution

$$p = u \frac{p'^{17/16}}{p_1'^{1/16}} \quad . \quad . \quad . \quad . \quad (40)$$

where

$$p_1' = \tfrac{1}{3}aT_1^4 \quad . \quad . \quad . \quad . \quad (41)$$

Then (39) transforms into

$$\frac{p_1'^{1/16}}{p'^{1/16}} + p' \frac{du}{dp'} + \frac{17}{16} u = \frac{cg}{F\alpha} \frac{1 + \bar{x}}{\bar{x}} \frac{p_1'^{1/8}}{(\frac{1}{3}a)^{9/8}} \frac{1}{u} . \quad . \quad (42)$$

Denote the coefficient of $u^{-1}$ on the right-hand side by A. The term $p_1'^{1/16}/p'^{1/16}$ is equal to $T_1^{\frac{1}{4}}/T^{\frac{1}{4}}$, which varies from $2^{1/16}$ to unity between the boundary and the photosphere. We shall neglect this exceedingly small variation and replace it by unity. We then have

$$1 + p' \frac{du}{dp'} + \frac{17}{16} u = \frac{A}{u}$$

or

$$\frac{dp'}{p'} = \frac{u\,du}{A - u - \frac{17}{16}u^2} \quad . \quad . \quad . \quad (43)$$

Writing

$$A - u - \tfrac{17}{16}u^2 \equiv -\tfrac{17}{16}(u - u_1)(u + u_2) \quad . \quad . \quad (44)$$

* The calculation of the value of $\kappa$ at the photosphere depends on a particular theory of $\kappa$. The analysis could be modified to suit any other theory.

† I have not succeeded in solving the equation when $x$, the ionization, is taken to vary with the depth.

we find on integration

$$\log \frac{p'}{p'_0} = -\frac{16/17}{u_1+u_2}\left[u_1 \log\left(1-\frac{u}{u_1}\right) + u_2 \log\left(1+\frac{u}{u_2}\right)\right] \qquad (45)$$

since $u = 0$ when $p' = p'_0 = \frac{1}{3}aT_0^4$. This equation shows that as $p'$ increases with depth, $u$ rapidly increases and tends to $u_1$.

The parameter $u$ will not quite have reached the value $u_1$ at the photospheric level $T = T_1$. Let us suppose at this depth it has reached the value $\psi u_1$. (We evaluate $\psi$ below.) Applying (40) to the photospheric level $p' = p_1'$ we have

$$p_1 = \psi u_1 p_1'$$

or

$$\frac{\beta_1}{1-\beta_1} = \psi u_1 \qquad (46)$$

But by (44)

$$A - u_1 - \tfrac{1}{16}u_1^2 = 0.$$

Hence by (46)

$$A - \frac{\beta_1}{1-\beta_1}\frac{1}{\psi} - \frac{17}{16}\frac{\beta_1^2}{(1-\beta_1)^2}\frac{1}{\psi^2} = 0$$

or

$$\frac{\beta_1}{(1-\beta_1)^2\psi}[1+\beta_1(1\tfrac{7}{16}\psi^{-1}-1)] = A \qquad (47)$$

Now

$$\frac{g}{F} = \frac{GM}{r_1^2}\cdot\frac{4\pi r_1^2}{L} = \frac{4\pi GM}{L} \qquad (48)$$

Hence by (42)

$$A = \frac{4\pi cGM}{\alpha L}\frac{1+\bar{x}}{\bar{x}}\frac{T_1^{\frac{1}{2}}}{\frac{1}{3}a} \qquad (49)$$

Inserting this in (47) and solving for L we have

$$L = \frac{4\pi cGM}{\frac{1}{3}a\alpha}\frac{1+\bar{x}}{\bar{x}}T_1^{\frac{1}{2}}\frac{(1-\beta_1)^2}{\beta_1}\frac{\psi}{1+\beta_1(1\tfrac{7}{16}\psi^{-1}-1)} \qquad (50)$$

Equation (50) is the four-term relation between L, M, the opacity constant $\alpha$ and C (involved in $\beta_1$ by (24)) when the precise boundary conditions are taken into account. Let us compare it with what we get by using our cruder formula (28). The value of $\kappa_1$, by (36) and (37), is given by

$$\kappa_1 = \frac{\alpha\bar{x}}{1+\bar{x}}\cdot\frac{p_1}{\frac{1}{3}aT_1^4}\cdot\frac{\frac{1}{3}a}{T_1^{\frac{1}{2}}} = \frac{\alpha\bar{x}}{1+\bar{x}}\frac{\frac{1}{3}a}{T_1^{\frac{1}{2}}}\frac{\beta_1}{1-\beta_1} \qquad (51)$$

Inserting this in (28) we get

$$L = \frac{4\pi cGM}{\frac{1}{3}a\alpha}\frac{1+\bar{x}}{\bar{x}}T_1^{\frac{1}{2}}\frac{(1-\beta_1)^2}{\beta_1} \qquad (50')$$

The only difference between (50) and (50′) is in the coefficient involving $\psi$. Since $\psi$ is nearly unity this coefficient is nearly unity.

This discrepancy simply represents the imperfection of our fit of the layers outside $T = T_1$ on to the layers inside $T_1$. It is not certain that our approximations are trustworthy to the extent of retaining $\psi$, but it is useful to find the size of the correction.

21. *Determination of $\psi$ and the $\psi$-Factor.*—From (44) we have

$$\tfrac{17}{16}(u_2 - u_1) = 1.$$

Substitute this value of $u_2$ in (45) and then apply (45) to the photospheric level. At this level $p' = p_1' = 2p_0'$ and $u = \psi u_1$. Hence

$$\log 2 = \frac{-16/17}{2u_1 + 16/17}\left[u_1 \log(1-\psi) + (16/17 + u_1)\log\left(1 + \frac{\psi u_1}{16/17 + u_1}\right)\right].$$

Now express $u_1$ in terms of $\beta_1$ by means of (46). After some reduction we find

$$1 - \psi = (\tfrac{1}{2})^{\frac{17}{8}+\frac{\psi(1-\beta_1)}{\beta_1}}\left[\frac{\psi^{-1} + \frac{16}{17}\frac{1-\beta_1}{\beta_1}}{\psi^{-1} + 1 + \frac{16}{17}\frac{1-\beta_1}{\beta_1}}\right]^{1+\frac{16}{17}\frac{\psi(1-\beta_1)}{\beta_1}} \quad . \quad (52)$$

This equation determines $\psi$ in terms of $\beta_1$. It must be solved by trial. The following short table is calculated from (52).

TABLE I.

| $1-\beta_1$. | $\psi$. | $\psi/[1+\beta_1(\frac{1}{8}\psi^{-1}-1)]$. |
|---|---|---|
| 0·0 | 0·878 | 0·725 |
| 0·1 | 0·888 | 0·754 |
| 0·2 | 0·900 | 0·786 |
| 0·3 | 0·913 | 0·819 |
| 0·4 | 0·928 | 0·853 |
| 0·5 | 0·944 | 0·888 |
| 0·7 | 0·980 | 0·956 |
| 1·0 | 1·000 | 1·000 |

22. *Determination of the Ionization Factor.*—Before (50) or (50′) becomes an explicit formula we have to show how to calculate the factor $(1 + \bar{x})/\bar{x}$ at the photospheric level. This depends on an estimate of the significant ionization-potentials. We shall suppose for definiteness that the material has a first stage ionization-potential of $\chi_1 = 6{\cdot}08$ volts and a second-stage ionization-potential of $\chi_2 = 11{\cdot}82$ volts. These are the values for calcium, but of course this does not mean we are assuming the photospheric layers to be entirely composed of calcium. But it is particularly desirable to avoid the appearance of choosing I.P.'s to suit the later results. All we want are some ionization-potentials that determine the abundance of free electrons.

First-stage ionization will go on with increasing temperature until it is practically complete. Thereafter second-stage ionization will set

in. The two ionization-potentials $\chi_1$ and $\chi_2$ are sufficiently far apart for us to be able to calculate the two stages separately.

For the first stage we have for the ionization at the photosphere $T = T_1$,

$$\frac{\bar{x}}{1-\bar{x}}(P_e)_1 = \frac{2(2\pi m)^{\frac{3}{2}}k^{\frac{5}{2}}}{h^3}\frac{q_1}{q_0}T_1^{\frac{5}{2}}e^{-\chi_1/kT_1} = K_1, \quad . \quad (53)$$

say, in a well-known notation.*

Now,

$$\frac{(P_e)_1}{p_1} = \frac{\bar{x}}{1+\bar{x}}, \qquad \frac{p_1}{p_1'} = \frac{\beta_1}{1-\beta_1}.$$

Eliminating $(P_e)_1$ and $p_1$ we have from (53)

$$\frac{\bar{x}^2}{1-\bar{x}^2} = \frac{1-\beta_1}{\beta_1}\frac{K_1}{p_1'},$$

whence

$$\bar{x} = \frac{1}{(1+d_1)^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad (54)$$

where

$$d_1 = \frac{\beta_1}{1-\beta_1}\frac{p_1'}{K_1} \quad . \quad . \quad . \quad . \quad (54')$$

Since $p_1' = \frac{1}{3}aT_1^4$, $d_1$ is a function of $T_1$ and $\beta_1$ only, and hence so is $\bar{x}$.

For the second stage we have, putting $\bar{x} = 1 + \bar{y}$,

$$\frac{\bar{y}}{1-\bar{y}}(P_e)_1 = \frac{2(2\pi m)^{\frac{3}{2}}k^{\frac{5}{2}}}{h^3}\frac{q_2}{q_1}T_1^{\frac{5}{2}}e^{-\chi_2/kT_2} = K_2 . \quad . \quad (53')$$

say. Also

$$\frac{(P_e)_1}{p_1} = \frac{\bar{x}}{1+\bar{x}}, \qquad \frac{p_1}{p_1'} = \frac{\beta_1}{1-\beta_1},$$

whence

$$\bar{x} = \frac{1}{2} + \left[\frac{1}{4} + \frac{2}{1+d_2}\right]^{\frac{1}{2}} \quad . \quad . \quad . \quad (55)$$

where

$$d_2 = \frac{\beta_1}{1-\beta_1}\frac{p_1'}{K_2} \quad . \quad . \quad . \quad . \quad (55')$$

As before, $\bar{x}$ is a function of $T_1$ and $\beta_1$ only. Completion of first-stage ionization corresponds to $d_1$ very small, $d_2$ very large, for which values it will be observed that (54) and (55) both reduce to $\bar{x} = 1$. As $d_2$ decreases with increasing temperature, $\bar{x} \to 2$.

The numerical values of the " weight-factors " for calcium are

$$q_0 = 1, \qquad q_1 = 2, \qquad q_2 = 1,$$

and

$$\frac{2(2\pi m)^{3/2}k^{5/2}}{h^3} = 0{\cdot}664.$$

* R. H. Fowler, *Statistical Mechanics*, p. 370, equation (1072).

The following table gives the values of $\bar{x}$ calculated from these formulæ for given $1 - \beta_1$ and $T_1$.

TABLE II.

*Values of Ionization* $\bar{x}$ *(Number of Free Electrons per Atom) at the Photosphere, for Stars of given* $1-\beta_1$, *and Effective Temperature* $T_1$. *The Calculations have been carried through for a Gas having Mean First-stage and Second-stage Ionization-Potentials equal to those of Calcium.*

| $1-\beta_1$. | 3000°. | 3500°. | 4000°. | 5000°. | 6000°. | 7000°. | 8000°. | 9000°. | 10,000°. | 12,000°. | 14,000°. | 16,000°. | 18,000°. | 20,000°. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ·003 | ·0236 | ·1126 | ·340 | ·873 | 1·000 | 1·004 | 1·035 | 1·166 | 1·416 | 1·830 | 1·952 | 1·982 | 1·992 | 1·995 |
| ·006 | ·0334 | ·1588 | ·457 | ·930 | 1·000 | 1·008 | 1·069 | 1·277 | 1·580 | 1·905 | 1·976 | 1·990 | 1·996 | 1·998 |
| ·009 | ·0409 | ·1929 | ·532 | ·950 | 1·001 | 1·012 | 1·098 | 1·358 | 1·670 | 1·936 | 1·985 | 1·994 | 1·997 | 1·998 |
| ·012 | ·0473 | ·2218 | ·588 | ·963 | 1·001 | 1·016 | 1·126 | 1·423 | 1·730 | 1·951 | 1·988 | 1·995 | 1·998 | 1·999 |
| ·015 | ·0530 | ·2470 | ·631 | ·970 | 1·001 | 1·020 | 1·152 | 1·475 | 1·770 | 1·960 | 1·990 | 1·996 | 1·998 | 1·999 |
| ·020 | ·0613 | ·2835 | ·685 | ·976 | 1·001 | 1·027 | 1·191 | 1·545 | 1·816 | 1·970 | 1·992 | 1·997 | 1·999 | 1·999 |
| ·025 | ·0687 | ·313 | ·727 | ·981 | 1·002 | 1·034 | 1·226 | 1·598 | 1·847 | 1·975 | 1·994 | 1·998 | 1·999 | 1·999 |
| ·030 | ·0755 | ·342 | ·757 | ·985 | 1·002 | 1·041 | 1·259 | 1·640 | 1·870 | 1·980 | 1·995 | 1·998 | 1·999 | 2·0 |
| ·040 | ·0875 | ·388 | ·803 | ·989 | 1·003 | 1·051 | 1·322 | 1·703 | 1·899 | 1·985 | 1·996 | 1·999 | 1·999 | 2·0 |
| ·050 | ·0983 | ·427 | ·834 | ·991 | 1·003 | 1·063 | 1·364 | 1·625 | 1·920 | 1·988 | 1·997 | 1·999 | 2·0 | 2·0 |
| ·060 | ·1082 | ·461 | ·857 | ·993 | 1·004 | 1·075 | 1·406 | 1·781 | 1·932 | 1·990 | 1·998 | 1·999 | 2·0 | 2·0 |
| ·070 | ·1173 | ·493 | ·875 | ·994 | 1·005 | 1·087 | 1·444 | 1·807 | 1·946 | 1·991 | 1·998 | 1·999 | 2·0 | 2·0 |
| ·080 | ·1260 | ·520 | ·889 | ·995 | 1·006 | 1·099 | 1·477 | 1·830 | 1·950 | 1·993 | 1·998 | 1·999 | 2·0 | 2·0 |
| ·090 | ·1341 | ·544 | ·900 | ·995 | 1·006 | 1·111 | 1·510 | 1·845 | 1·956 | 1·994 | 1·999 | 1·999 | 2·0 | 2·0 |
| ·100 | ·1419 | ·567 | ·909 | ·996 | 1·007 | 1·122 | 1·536 | 1·860 | 1·960 | 1·995 | 1·999 | 1·999 | 2·0 | 2·0 |
| ·120 | ·1571 | ·605 | ·923 | ·996 | 1·009 | 1·145 | 1·612 | 1·882 | 1·967 | 1·996 | 1·999 | 1·999 | 2·0 | 2·0 |
| ·140 | ·1710 | ·640 | ·935 | ·997 | 1·010 | 1·167 | 1·625 | 1·899 | 1·973 | 1·996 | 1·999 | 2·0 | 2·0 | 2·0 |
| ·160 | ·1845 | ·667 | ·942 | ·997 | 1·012 | 1·188 | 1·660 | 1·911 | 1·977 | 1·997 | 1·999 | 2·0 | 2·0 | 2·0 |
| ·180 | ·1976 | ·694 | ·952 | ·997 | 1·014 | 1·210 | 1·689 | 1·923 | 1·980 | 1·997 | 1·999 | 2·0 | 2·0 | 2·0 |
| ·200 | ·2100 | ·718 | ·957 | ·997 | 1·016 | 1·231 | 1·715 | 1·932 | 1·982 | 1·997 | 1·999 | 2·0 | 2·0 | 2·0 |
| ·220 | ·2228 | ·739 | ·960 | ·998 | 1·018 | 1·20 | 1·738 | 1·940 | 1·983 | 1·998 | 1·999 | 2·0 | 2·0 | 2·0 |
| ·240 | ·2348 | ·759 | ·965 | ·998 | 1·020 | 1·272 | 1·760 | 1·945 | 1·985 | 1·998 | 1·999 | 2·0 | 2·0 | 2·0 |
| ·260 | ·2472 | ·774 | ·968 | ·998 | 1·022 | 1·292 | 1·778 | 1·950 | 1·986 | 1·998 | 1·999 | 2·0 | 2·0 | 2·0 |
| ·280 | ·2590 | ·788 | ·971 | ·998 | 1·023 | 1·312 | 1·792 | 1·953 | 1·988 | 1·998 | 1·999 | 2·0 | 2·0 | 2·0 |
| ·300 | ·2712 | ·803 | ·973 | ·998 | 1·026 | 1·332 | 1·809 | 1·956 | 1·989 | 1·999 | 1·999 | 2·0 | 2·0 | 2·0 |
| ·320 | ·2824 | ·816 | ·975 | ·999 | 1·028 | 1·340 | 1·822 | 1·960 | 1·990 | 1·999 | 1·999 | 2·0 | 2·0 | 2·0 |
| ·340 | ·2950 | ·828 | ·977 | ·999 | 1·031 | 1·370 | 1·833 | 1·964 | 1·991 | 1·999 | 1·999 | 2·0 | 2·0 | 2·0 |
| ·360 | ·3078 | ·838 | ·980 | ·999 | 1·033 | 1·390 | 1·846 | 1·967 | 1·992 | 1·999 | 1·999 | 2·0 | 2·0 | 2·0 |
| ·380 | ·319 | ·849 | ·982 | ·999 | 1·036 | 1·408 | 1·857 | 1·969 | 1·992 | 1·999 | 1·999 | 2·0 | 2·0 | 2·0 |
| ·400 | ·331 | ·860 | ·983 | ·999 | 1·039 | 1·428 | 1·867 | 1·972 | 1·993 | 1·999 | 1·999 | 2·0 | 2·0 | 2·0 |
| ·420 | ·344 | ·868 | ·984 | 1·00 | 1·043 | 1·446 | 1·875 | 1·974 | 1·993 | 1·999 | 1·999 | 2·0 | 2·0 | 2·0 |

Table II.—*continued.*

| $1-\beta_1$. | 3000°. | 3500°. | 4000°. | 5000°. | 6000°. | 7000°. | 8000°. | 9000°. | 10,000°. | 12,000°. | 14,000°. | 16,000°. | 18,000°. | 20,000°. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ·440 | ·357 | ·877 | ·985 | 1·00 | 1·046 | 1·465 | 1·883 | 1·976 | 1·994 | 1·999 | 1·999 | 2·0 | 2·0 | 2·0 |
| ·460 | ·369 | ·885 | ·986 | 1·00 | 1·050 | 1·484 | 1·890 | 1·978 | 1·994 | 1·999 | 1·999 | 2·0 | 2·0 | 2·0 |
| ·480 | ·382 | ·892 | ·987 | 1·00 | 1·054 | 1·504 | 1·898 | 1·980 | 1·995 | 1·999 | 1·999 | 2·0 | 2·0 | 2·0 |
| ·500 | ·395 | ·899 | ·988 | 1·00 | 1·058 | 1·523 | 1·906 | 1·981 | 1·995 | 1·999 | 1·999 | 2·0 | 2·0 | 2·0 |
| ·540 | ·422 | ·912 | ·990 | 1·00 | 1·068 | 1·562 | 1·918 | 1·983 | 1·998 | 2·0 | 2·0 | 2·0 | 2·0 | 2·0 |
| ·580 | ·450 | ·923 | ·992 | 1·00 | 1·077 | 1·599 | 1·930 | 1·987 | 1·999 | 2·0 | 2·0 | 2·0 | 2·0 | 2·0 |
| ·620 | ·480 | ·932 | ·993 | 1·00 | 1·092 | 1·635 | 1·940 | 1·990 | 1·999 | 2·0 | 2·0 | 2·0 | 2·0 | 2·0 |
| ·660 | ·513 | ·943 | ·994 | 1·00 | 1·105 | 1·673 | 1·950 | 1·991 | 1·999 | 2·0 | 2·0 | 2·0 | 2·0 | 2·0 |
| ·700 | ·550 | ·952 | ·995 | 1·00 | 1·122 | 1·710 | 1·960 | 1·994 | 2·0 | 2·0 | 2·0 | 2·0 | 2·0 | 2·0 |

23. We now collect together the significant formulæ:

$$L = \frac{4\pi cGM}{\frac{1}{3}a\alpha}\,\frac{1+\bar{x}}{\bar{x}}\,T_1^{\frac{1}{2}}\,\frac{(1-\beta_1)^2}{\beta_1} \quad . \quad . \quad . \quad (50')$$

$$\frac{1-\beta_1}{\beta_1^{\frac{4}{3}}}\left(\frac{R}{\mu_1}\right)^4 = \frac{\pi a G^3 M^2}{12}\,C \quad . \quad . \quad . \quad (24)$$

$\bar{x}$ is given by the preceding table as a function of $1-\beta_1$ and $T_1$. When accurate work is required, (50) is to be preferred to (50′). The coefficient C is given by

$$C = \frac{\phi_1}{x_1^4 (d\theta/dx)_1^2} \quad . \quad . \quad . \quad . \quad (25)$$

where $\phi(\theta)$ is a function of $\theta$ equal to 1 when $\theta = 1$ and tending to a finite limit $\phi_1$, as $\theta \to 0$, and $\theta$ is a function of $x$ equal to 1 at $x = 0$ and equal to 0 at $x = x_1$ given by the differential equation

$$\frac{4}{x^2}\frac{d}{dx}\left[x^2\frac{d\theta}{dx}\{\phi(\theta) + \tfrac{1}{4}\theta\phi'(\theta)\}\right] = -\theta^3 \quad . \quad . \quad (20)$$

with the condition $d\theta/dx = 0$ at $x = 0$.

These amount to a four-term relation, true for any star, connecting L, M, $T_1$ and the coefficient C.†

When $\phi$ is constant and equal to unity (20) reduces to Emden's differential equation for the polytrope $n = 3$, and the coefficient C takes the value of Eddington's constant; (24) takes the form of Eddington's "quartic equation" for $\beta$, except that $\beta_1$ and $\mu_1$ refer to the photosphere; and (50′) is always equivalent (whatever is the form of $\phi$) to

$$L = \frac{4\pi cGM(1-\beta_1)}{\kappa_1} \quad . \quad . \quad . \quad (28)$$

† We have replaced $\kappa_1$ in terms of a universal constant $\alpha$ and $T_1$, hence the mention of $T_1$ in the four-term relation as compared with (L, M, $\kappa_1$, C).

When $\phi = 1$, $(1 - \beta)/\beta^4\mu^4$ is constant if the star is gaseous, and if $\mu$ is constant, so is $\beta$; (28) then reduces to the form deduced by Eddington on his approximation "$\kappa\eta$ = constant" on putting $\eta = 1$, its value at the photosphere. Thus our equations are straightforward generalisations of Eddington's equations, reducing to them when his assumptions are satisfied.

It is known from opacity-values provided by spectroscopic data † that the value $1 - \beta_1$ at the photosphere is not widely different from the value calculated by Eddington for the interior on the hypothesis "$\kappa\eta$ = constant." Apart from variations of $\mu$, therefore, $\phi$ would not be expected to depart widely from unity. Actually variations of $\mu$ must be important near the photosphere, so that we cannot give C its Emden value. It has to be found from observation for each star separately and we have no reason to expect it to be the same for all stars.

It may be repeated at this stage that there should be nothing surprising in the appearance of the *photospheric* opacity in these formulæ. For given $\phi$, the star is in neutral equilibrium mechanically, *i.e.* the acceleration of each particle is instantaneously zero. If L is given as well, the radius, and so the effective temperature, must adjust themselves until $\kappa_1$ has the right value for (28) to be satisfied. If this adjustment is not perfect, the star will cool faster or slower than it is being provided with energy L, the elements will contract or expand, and the configuration will be gently propelled towards one in which (28) is satisfied.

## III. *Application to Actual Stars.*

24. *Construction of a Homologous Family.*—As explained in the introduction, we can in principle, for a star of observed L, M, and $T_1$, compute $1 - \beta_1$ from (50′) from a knowledge of $\alpha$. Then from (24) we could compute C, and so form an estimate of the way $\phi(\theta)$ must depart from unity. Owing to the intricate way in which $\beta_1$ enters into the factor $(1 + \bar{\alpha})/\bar{\alpha}$, this would be very complicated; further we have no really accurate knowledge of $\alpha$. Instead we propose to construct one definite homologous family (*i.e.* choose some definite $\phi$) and use it as a standard of comparison. Actually we need not the whole form of $\phi$ but only the single coefficient C dependent on it. We construct this family by choosing the family which contains two stars Sun* and Capella* indistinguishable observationally from Sun and Capella.

For Sun* and Capella* the coefficient C has the same value. Further, the ionization is the same in the atmospheres of Sun* and Capella*, since they have the same spectrum. The photospheric molecular weight $\mu_1$ may also be taken the same for the two stars. Hence equations (50′) and (24) applied to these two stars become

$$1 - \beta_1 \propto \beta_1^4 M^2$$

$$L \propto M \frac{(1 - \beta_1)^2}{\beta_1} T_1^{\frac{1}{2}},$$

† See Bakerian Lecture, 1929.

whence

$$L \propto M^{\frac{3}{2}}(1-\beta_1)^{\frac{7}{4}}T_1^{\frac{1}{2}}.$$

Hence, if unprimed symbols refer to Sun*, primed symbols to Capella*, we have

$$\frac{L'}{L} = \left(\frac{1-\beta_1'}{1-\beta_1}\right)^{\frac{7}{4}}\left(\frac{M'}{M}\right)^{\frac{3}{2}}\left(\frac{T_1'}{T_1}\right)^{\frac{1}{2}} \quad . \quad . \quad (56)$$

$$\frac{1-\beta_1'}{1-\beta_1} = \frac{\beta_1'^4}{\beta_1^4}\frac{M'^2}{M^2} \quad . \quad . \quad . \quad . \quad (57)$$

From the values quoted by Eddington we have

$$L'/L = 126, \qquad M'/M = 4{\cdot}18, \qquad T_1'/T_1 = 5200/5740.$$

We find, then, from (56)

$$\frac{1-\beta_1'}{1-\beta_1} = 4{\cdot}787,$$

whence from (57)

$$\frac{\beta_1'}{\beta_1} = 0{\cdot}7235.$$

Solving these two equations we find

$$1-\beta_1 = 0{\cdot}06804,$$
$$1-\beta_1' = 0{\cdot}3257.$$

The quartic relation between $\beta_1$ and M for these two stars then takes the form

$$1-\beta_1 = 0{\cdot}0902\beta_1^4(M/M_\odot)^2 \quad . \quad . \quad . \quad (58)$$

From an estimated value of $\mu_1$ and the known values of $a$ and G we could now deduce the value of C. If, however, all members of our homologous family have the same photospheric molecular weight $\mu_1$, there is no need to compute C. Equation (58) then determines $1-\beta_1$ for given M for all members of the homologous family. For want of real knowledge of the mean molecular weight in the photospheres of actual stars we shall adopt the simple assumption that $\mu_1$ is the same for all stars of our homologous family.†

We now compute the photospheric opacity of our homologous family. We use formula (50) in the form

$$L = \frac{4\pi cGM(1-\beta_1)}{\kappa_1}\frac{\psi}{1+\beta_1(\frac{17}{16}\psi^{-1}-1)},$$

which it takes in virtue of (51). For simplicity we have taken $\psi = 0{\cdot}9$ instead of the actual values for the corresponding $1-\beta_1$'s. We have used for $L_\odot$ the value $3{\cdot}780 \times 10^{33}$ ergs sec.$^{-1}$ and for $M_\odot$ the value

† We have earlier apparently assumed the photospheric material to be made of calcium, but this was only to get the abundance of free electrons. Suggestions are now being made that the atmospheres of the actual stars contain a large proportion of hydrogen. This would seriously affect $\mu$, but not seriously the electron abundance.

$1{\cdot}985 \times 10^{33}$ grams given in the appendix to Eddington's *Internal Constitution of the Stars*. The following are the results :—

| | *Sun**. | *Capella**. |
|---|---|---|
| M | $=1{\cdot}9{\cdot}85\times10^{33}$ grams | $4{\cdot}18\times1{\cdot}985\times10^{33}$ grams |
| L | $=3{\cdot}780\times10^{33}$ ergs sec.$^{-1}$ | $126\times3{\cdot}780\times10^{33}$ ergs sec.$^{-1}$ |
| $1-\beta_1$ | $=0{\cdot}06804$ | $0{\cdot}3257$ |
| $\kappa_1$ | $=708$ gram$^{-1}$ cm.$^{-2}$ | $112$ gram$^{-1}$ cm.$^{-2}$ |

The gas-pressures to which these absorption coefficients refer are given, for each star, by

$$p_1 = \frac{\beta_1}{1-\beta_1}p_1' = \frac{\beta_1}{1-\beta_1}\tfrac{1}{3}aT_1^4,$$

with

$$T_1 = 5740^\circ, \qquad T_1' = 5200^\circ.$$

We find

| | *Sun**. | *Capella**. |
|---|---|---|
| $p_1$ | $=$ 37·9 dynes cm.$^{-2}$ | 3·85 dynes cm.$^{-2}$ |

For the I.P.'s assumed, the matter at these pressures and temperatures is just once-ionized. Hence $\bar{x} = 1$ and $(P_e)_1 = \frac{1}{2}p_1$. The electron-pressures are therefore

| | *Sun**. | *Capella**. |
|---|---|---|
| $(P_e)_1$ | $=$ 18·9 dynes cm.$^{-2}$ | 1·92 dynes cm.$^{-2}$ |

The value of $a$ may be computed either directly from $\kappa_1 = a(P_e)_1/T_1^{9/2}$ or from (50). The two methods are simply different ways of arranging the arithmetic. The results are

| | | *Sun**. | *Capella**. |
|---|---|---|---|
| $a\times10^{-18}$ | $= \dfrac{\kappa_1}{(P_e)_1}\left(\dfrac{T_1}{10^4}\right)^{9/2}$ | $=$ 3·07 | 3·07 |

The agreement of the two values of $a$ from Sun* and Capella* simply checks our arithmetic—we have already taken $a$ to be a universal constant, and therefore the same for our two special stars.

The above results refer to the photospheric level $T = T_1$, where by the formula $T^4 = \frac{1}{2}T_1^4(1 + \frac{3}{4}\tau)$ the optical depth is $\tau = \frac{2}{3}$. Let us compare the foregoing values for Sun* and Capella* with the actual values for Sun and Capella as determined from spectroscopic data. In Bakerian Lecture, 1929, I have given (Tables IX. and X.) tables of the pressures and coefficients of opacity at various optical depths in the Sun and Capella computed solely from spectroscopic data (a certain zinc line maximum). Interpolating in these tables for $\tau = \frac{2}{3}$ we find values as follows :—

| | *Sun.* | *Capella.* |
|---|---|---|
| $p_1$ | 70·6 dynes cm.$^{-2}$ | 8·0 dynes cm.$^{-2}$ |
| $\kappa_1$ | 363 grams$^{-1}$ cm.$^{-2}$ | 64·7 grams$^{-1}$ cm.$^{-2}$ |

The value of $a \times 10^{-18}$ found by the spectroscopic method was 0·85.

The comparison is striking. Sun* and Capella* have opacities about twice those of Sun and Capella; and the levels at which the methods pick out the photosphere are at pressures for Sun* and Capella* about half those for Sun and Capella; the constant $\alpha$ is in consequence about 4 times as big for Sun* and Capella* as for Sun and Capella.

Considering the uncertainties in the spectroscopic method we may say that the agreement is practically complete. What does this agreement signify? We have urged no reason why Sun and Capella should belong to the same homologous family.† But the numerical agreements show that if we had reversed our procedure and used the spectroscopic determination of $\alpha$ to compute the values C and C′ from the observed (L, M, $T_1$) and (L′, M′, $T_1'$) we should have found that C and C′ came out practically the same number. In other words, such evidence as the value of C provides as to the form of the function $\phi(\theta)$ is that $\phi$ is the same for the actual stars Sun and Capella.

I wish to emphasize that this is an observational result; whether one would have expected it *a priori* is quite another matter which I see little value in discussing. All we have worked into our analysis are the assumptions (1) that the stars are in mechanical equilibrium; (2) and (3) that their outer layers are gaseous and in radiative equilibrium; (4) the usual theory of photospheric opacity; (5) an assumption about the state of ionization in the photospheres. All these assumptions rest on direct observation; in particular (4) in its dependence on electron-pressure is confirmed by observed absolute magnitude effects. We have not assumed the interiors to be gaseous or to be in radiative equilibrium.

This agreement of the opacities shows us that Sun* and Capella* require to be composed of material closely the same as the actual material of which Sun and Capella are composed. We now suppose the other members of the homologous family are composed of the same material (same $\alpha$) and construct a table giving their luminosities for given mass and effective temperature. (Inside a *given family* the four-term relation (L, M, $T_1$, C) becomes a three-term relation (L, M, $T_1$), since C is the same for all members of the family.)

To calculate the luminosity of a member of our homologous family it is simplest to use Sun* as a standard. We then have

$$1 - \beta_1 = 0{\cdot}0902\beta_1^4(M/M_\odot)^2 \quad . \quad . \quad . \quad (58)$$

$$\frac{L}{L_\odot} = \frac{M}{M_\odot}\frac{1+\bar{x}}{2\bar{x}}\left(\frac{1-\beta_1}{1-\beta_{1\odot}}\right)^2\frac{\beta_{1\odot}}{\beta_1}\left(\frac{T_1}{T_{1\odot}}\right)^{\frac{1}{2}} \quad . \quad . \quad (59)$$

with

$$1 - \beta_{1\odot} = 0{\cdot}06804, \qquad T_{1\odot} = 5740°.$$

† Though they are obviously likely to do so on general grounds of similarity. If it could be *assumed* on these grounds that their density distributions were similar our analysis would afford a determination of photospheric opacity from observed masses, luminosities, and temperatures. But I have carefully avoided this assumption.

The form of the ionization factor arises from the fact that $[(1 + \bar{x})/\bar{x}]_{\odot} = 2$. To derive the absolute bolometric magnitude $m$ we use the formula

$$m = 4{\cdot}85 - 2{\cdot}5 \log_{10} (L/L_{\odot}) \quad . \quad . \quad . \quad (60)$$

Tables III. and IV. give the results. The procedure is to start with a value of $1 - \beta_1$, and compute M from (58). Then derive L from (59), using the previously given table (Table II.) to give the ionization factor.

TABLE III.

*Values of* $1-\beta_1$ *for given Mass* M, *for a certain Homologous Family.*

| M. | $1-\beta_1$. | M. | $1-\beta_1$. | M. | $1-\beta_1$. |
|---|---|---|---|---|---|
| 0·0 | ·00 | 2·2 | ·1889 | 7·0 | ·4387 |
| 0·2 | ·0036 | 2·4 | ·2063 | 7·5 | ·4533 |
| 0·3 | ·0079 | 2·6 | ·2227 | 8·0 | ·4668 |
| 0·4 | ·0137 | 2·8 | ·2381 | 9·0 | ·4909 |
| 0·5 | ·0210 | 3·0 | ·2529 | 10·0 | ·5119 |
| 0·6 | ·0289 | 3·2 | ·2669 | 11·0 | ·5305 |
| 0·7 | ·0379 | 3·4 | ·2801 | 12·0 | ·5470 |
| 0·8 | ·0475 | 3·6 | ·2927 | 13·0 | ·5618 |
| 0·9 | ·0576 | 3·8 | ·3046 | 14·0 | ·5753 |
| 1·0 | ·0680 | 4·0 | ·3160 | 15·0 | ·5875 |
| 1·2 | ·0893 | 4·5 | ·3421 | 16·0 | ·5987 |
| 1·4 | ·1106 | 5·0 | ·3655 | 18·0 | ·6185 |
| 1·6 | ·1314 | 5·5 | ·3865 | 20·0 | ·6357 |
| 1·8 | ·1515 | 6·0 | ·4055 | | |
| 2·0 | ·1707 | 6·5 | ·4229 | | |

If accurate work is required the luminosities of Table IV. need small corrections for the value of the factor involving $\psi$. Table V. (p. 46), derived from Table I., gives the correction in magnitudes, the zero having been adjusted to make zero correction for the Sun, since this has been taken as standard. The corrections are to be *added* to the bolometric magnitudes of Table IV.

[TABLE

## Table IV.

*Absolute Bolometric Magnitudes for Stars of given Mass* M *and given Effective Temperature, from Formulæ* (58) *and* (59) *for a certain Homologous Family.*

| $1-\beta_1$. | M. | 3000°. | 3500°. | 4000°. | 5000°. | 6000°. | 7000°. | 8000°. | 9000°. | 10,000°. | 12,000°. | 14,000°. | 16,000°. | 18,000°. | 20,000°. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ·003 | 0·183 | 10·56 | 12·06 | 13·00 | 13·54 | 13·52 | 13·44 | 13·38 | 13·38 | 13·42 | 13·32 | 13·37 | 13·30 | 13·23 | 13·18 |
| ·006 | 0·261 | 9·03 | 10·52 | 11·35 | 11·69 | 11·63 | 11·55 | 11·51 | 11·53 | 11·57 | 11·54 | 11·47 | 11·40 | 11·34 | 11·28 |
| ·009 | 0·322 | 8·13 | 9·58 | 10·34 | 10·59 | 10·52 | 10·44 | 10·41 | 10·45 | 10·48 | 10·44 | 10·37 | 10·29 | 10·23 | 10·17 |
| ·012 | 0·374 | 7·49 | 8·92 | 9·62 | 9·81 | 9·73 | 9·65 | 9·63 | 9·68 | 9·71 | 9·65 | 9·58 | 9·50 | 9·44 | 9·38 |
| ·015 | 0·421 | 6·99 | 8·40 | 9·05 | 9·19 | 9·11 | 9·04 | 9·03 | 9·08 | 9·10 | 9·04 | 8·96 | 8·89 | 8·83 | 8·77 |
| ·020 | 0·490 | 6·34 | 7·72 | 8·31 | 8·40 | 8·31 | 8·24 | 8·25 | 8·30 | 8·31 | 8·24 | 8·17 | 8·09 | 8·03 | 7·97 |
| ·025 | 0·554 | 5·84 | 7·27 | 7·72 | 7·78 | 7·69 | 7·62 | 7·64 | 7·70 | 7·70 | 7·64 | 7·54 | 7·47 | 7·40 | 7·35 |
| ·030 | 0·613 | 5·42 | 6·74 | 7·23 | 7·27 | 7·18 | 7·11 | 7·14 | 7·19 | 7·19 | 7·11 | 7·03 | 6·96 | 6·89 | 6·84 |
| ·040 | 0·722 | 4·76 | 6·00 | 6·46 | 6·45 | 6·37 | 6·31 | 6·35 | 6·40 | 6·38 | 6·30 | 6·21 | 6·14 | 6·08 | 6·02 |
| ·050 | 0·825 | 4·21 | 5·46 | 5·84 | 5·82 | 5·73 | 5·67 | 5·74 | 5·74 | 5·75 | 5·66 | 5·58 | 5·50 | 5·44 | 5·38 |
| ·060 | 0·923 | 3·80 | 4·99 | 5·33 | 5·29 | 5·20 | 5·15 | 5·21 | 5·24 | 5·22 | 5·13 | 5·05 | 4·97 | 4·91 | 4·85 |
| ·070 | 1·018 | 3·42 | 4·58 | 4·89 | 4·84 | 4·74 | 4·70 | 4·77 | 4·80 | 4·77 | 4·68 | 4·59 | 4·52 | 4·46 | 4·40 |
| ·080 | 1·112 | 3·09 | 4·23 | 4·50 | 4·44 | 4·35 | 4·31 | 4·38 | 4·40 | 4·37 | 4·28 | 4·20 | 4·12 | 4·06 | 4·00 |
| ·090 | 1·206 | 2·80 | 3·90 | 4·15 | 4·09 | 3·99 | 3·95 | 4·03 | 4·05 | 4·02 | 3·91 | 3·84 | 3·77 | 3·71 | 3·65 |
| ·100 | 1·303 | 2·53 | 3·60 | 3·83 | 3·76 | 3·67 | 3·64 | 3·72 | 3·73 | 3·69 | 3·60 | 3·51 | 3·44 | 3·38 | 3·32 |
| ·120 | 1·489 | 2·06 | 3·08 | 3·27 | 3·20 | 3·10 | 3·09 | 3·17 | 3·17 | 3·13 | 3·07 | 2·95 | 2·88 | 2·81 | 2·76 |
| ·140 | 1·684 | 1·64 | 2·63 | 2·79 | 2·70 | 2·61 | 2·60 | 2·68 | 2·68 | 2·64 | 2·54 | 2·46 | 2·38 | 2·32 | 2·26 |
| ·160 | 1·887 | 1·28 | 2·22 | 2·36 | 2·26 | 2·17 | 2·17 | 2·25 | 2·25 | 2·20 | 2·10 | 2·02 | 1·94 | 1·88 | 1·82 |

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ·180 | 2·100 | 0·94 | 1·85 | 1·96 | 1·86 | 1·77 | 1·78 | 1·86 | 1·84 | 1·80 | 1·70 | 1·62 | 1·55 | 1·49 | 1·43 |
| ·200 | 2·326 | 0·63 | 1·50 | 1·60 | 1·50 | 1·41 | 1·42 | 1·50 | 1·48 | 1·44 | 1·34 | 1·25 | 1·18 | 1·12 | 1·06 |
| ·220 | 2·566 | 0·34 | 1·18 | 1·26 | 1·16 | 1·07 | 1·07 | 1·16 | 1·14 | 1·09 | 0·95 | 0·91 | 0·84 | 0·78 | 0·72 |
| ·240 | 2·823 | +0·07 | 0·87 | 0·94 | 0·84 | 0·75 | 0·78 | 0·85 | 0·82 | 0·77 | 0·68 | 0·59 | 0·52 | 0·46 | 0·40 |
| ·260 | 3·10 | −0·19 | 0·58 | 0·64 | 0·54 | 0·45 | 0·48 | 0·55 | 0·52 | 0·47 | 0·37 | 0·29 | 0·21 | 0·15 | 0·09 |
| ·280 | 3·40 | −0·44 | 0·30 | 0·35 | +0·24 | +0·16 | +0·20 | +0·26 | +0·23 | +0·18 | +0·08 | 0·00 | −0·08 | −0·14 | −0·20 |
| ·300 | 3·72 | −0·68 | +0·03 | +0·07 | −0·04 | −0·12 | −0·07 | −0·02 | −0·05 | −0·10 | −0·20 | −0·28 | −0·36 | −0·42 | −0·48 |
| ·320 | 4·07 | −0·92 | −0·23 | −0·28 | −0·30 | −0·39 | −0·34 | −0·28 | −0·32 | −0·37 | −0·47 | −0·55 | −0·62 | −0·69 | −0·75 |
| ·340 | 4·46 | −1·14 | −0·48 | −0·46 | −0·57 | −0·65 | −0·59 | −0·54 | −0·58 | −0·63 | −0·73 | −0·81 | −0·89 | −0·95 | −1·01 |
| ·360 | 4·88 | −1·36 | −0·73 | −0·71 | −0·82 | −0·90 | −0·84 | −0·79 | −0·84 | −0·89 | −0·98 | −1·07 | −1·14 | −1·20 | −1·26 |
| ·380 | 5·34 | −1·58 | −0·97 | −0·96 | −1·07 | −1·15 | −1·09 | −1·04 | −1·09 | −1·14 | −1·24 | −1·32 | −1·39 | −1·46 | −1·51 |
| ·400 | 5·85 | −1·80 | −1·21 | −1·21 | −1·32 | −1·40 | −1·32 | −1·29 | −1·33 | −1·38 | −1·48 | −1·56 | −1·64 | −1·70 | −1·76 |
| ·420 | 6·41 | −2·01 | −1·45 | −1·45 | −1·56 | −1·64 | −1·57 | −1·53 | −1·57 | −1·62 | −1·72 | −1·81 | −1·88 | −1·94 | −2·00 |
| ·440 | 7·04 | −2·22 | −1·68 | −1·69 | −1·79 | −1·88 | −1·80 | −1·77 | −1·81 | −1·86 | −1·97 | −2·05 | −2·12 | −2·18 | −2·24 |
| ·460 | 7·74 | −2·44 | −1·92 | −1·93 | −2·04 | −2·12 | −2·03 | −2·00 | −2·05 | −2·11 | −2·20 | −2·29 | −2·36 | −2·42 | −2·48 |
| ·480 | 8·69 | −2·65 | −2·15 | −2·17 | −2·28 | −2·35 | −2·26 | −2·24 | −2·29 | −2·34 | −2·44 | −2·53 | −2·60 | −2·66 | −2·72 |
| ·500 | 9·42 | −2·86 | −2·38 | −2·40 | −2·52 | −2·59 | −2·50 | −2·48 | −2·53 | −2·59 | −2·69 | −2·77 | −2·84 | −2·90 | −2·96 |
| ·540 | 11·56 | −3·29 | −2·86 | −2·88 | −3·00 | −3·06 | −2·97 | −2·95 | −3·01 | −3·06 | −3·16 | −3·24 | −3·32 | −3·38 | −3·44 |
| ·580 | 14·37 | −3·73 | −3·34 | −3·37 | −3·49 | −3·54 | −3·45 | −3·44 | −3·50 | −3·55 | −3·65 | −3·73 | −3·81 | −3·87 | −3·93 |
| ·620 | 18·15 | −4·19 | −3·84 | −3·88 | −3·99 | −4·05 | −3·94 | −3·95 | −4·00 | −4·06 | −4·16 | −4·24 | −4·31 | −4·38 | −4·44 |
| ·660 | 23·39 | −4·37 | −4·37 | −4·41 | −4·53 | −4·58 | −4·47 | −4·48 | −4·53 | −4·59 | −4·69 | −4·77 | −4·85 | −4·91 | −4·97 |
| ·700 | 30·95 | −5·19 | −4·93 | −4·98 | −5·09 | −5·13 | −5·02 | −5·04 | −5·10 | −5·16 | −5·26 | −5·34 | −5·42 | −5·48 | −5·55 |

*Note.*—The above magnitudes need small corrections as given in Table V. for the factor involving $\psi$.

TABLE V.

*Correction for $\psi$-factor.*

| $1-\beta_1$. | $\Delta m$. |
|---|---|
| 0·0 | +0·03 |
| 0·06804 | 0·00 |
| 0·1 | −0·01 |
| 0·2 | −0·06 |
| 0·3 | −0·10 |
| 0·4 | −0·15 |
| 0·5 | −0·19 |
| 0·7 | −0·27 |
| 1·0 | −0·32 |

25. Before comparing the luminosities of the actual stars as a whole with those of our homologous family, we will specially compute the luminosities of V-Puppis* and the two components of Krüger 60*. By V-Puppis* I mean a member of our homologous family which has the same mass M and effective temperature $T_1$ as the actual V-Puppis, and similarly for Krüger 60*.

For V-Puppis Eddington gives

$$M = 19{\cdot}2, \qquad T_1 = 19{,}000°.$$

From (58) we find

$$1 - \beta_1 = 0{\cdot}629.$$

For this star we may take the photospheric ionization to be $\bar{x} = 2{\cdot}0$. Formula (59) then gives

$$L/L_\odot = 5{\cdot}61 \times 10^3,$$

whence the uncorrected magnitude is − 4·52. The $\psi$-correction for $1 - \beta_1 = 0{\cdot}629$ is $\Delta m = -0{\cdot}24$, whence the absolute bolometric magnitude of V-Puppis* is

$$m = -4{\cdot}76.$$

From the observed radius (V-Puppis is an eclipsing variable) Eddington found the (observed) absolute bolometric magnitude of V-Puppis to be − 4·75. The value calculated by Eddington from his "mass-luminosity" law was − 5·06.

For Krüger 60* we consider the two components separately. According to Eddington the observed † data are :

| | | | |
|---|---|---|---|
| Bright component | M = 0·27 | $m = +9{\cdot}82$ | $T_1 = 3100°$. |
| Faint component | M = 0·16 | $m = +12{\cdot}32$ | |

† Actually only the combined mass is known, but I have taken over Eddington's distribution of mass between the two components. [Since the above computations were made, Sir James Jeans has drawn my attention to Aitken's determinations (*L.O.B.*, 365) which give masses 0·25 and 0·20. These would make Krüger 60*$b$ fainter and Krüger 60*$f$ brighter. A very small reduction in the assumed ionization-potentials would decrease both the calculated brightnesses.]

For Krüger 60**b* we find from (58)

$$1 - \beta_1 = 0{\cdot}0064.$$

From (53),

$$d_1 = 248,$$

whence from (54),

$$\bar{x} = 1/15{\cdot}80.$$

Hence

$$\frac{1 + \bar{x}}{2\bar{x}} = 8{\cdot}40.$$

Formula (59) then gives

$$L/L_\odot = 1{\cdot}386 \times 10^{-2},$$

whence the uncorrected bolometric magnitude is 9·50. The $\psi$-correction is $\Delta m = + 0{\cdot}03$, whence the absolute bolometric magnitude of Krüger 60**b* is

$$m = + 9{\cdot}53.$$

Eddington's computed value for Krüger 60*b* was + 11·07.

For Krüger 60**f* we find from (58),

$$1 - \beta_1 = 0{\cdot}00228.$$

From (53),

$$d_1 = 698,$$

whence from (54),

$$\bar{x} = 1/26{\cdot}4.$$

Hence

$$\frac{1 + \bar{x}}{2\bar{x}} = 13{\cdot}7.$$

Formula (59) then gives

$$L/L_\odot = 1{\cdot}636 \times 10^{-3},$$

whence the uncorrected bolometric magnitude is 11·82. The $\psi$-correction is $\Delta m = + 0{\cdot}03$, whence the absolute bolometric magnitude of Krüger 60**f* is

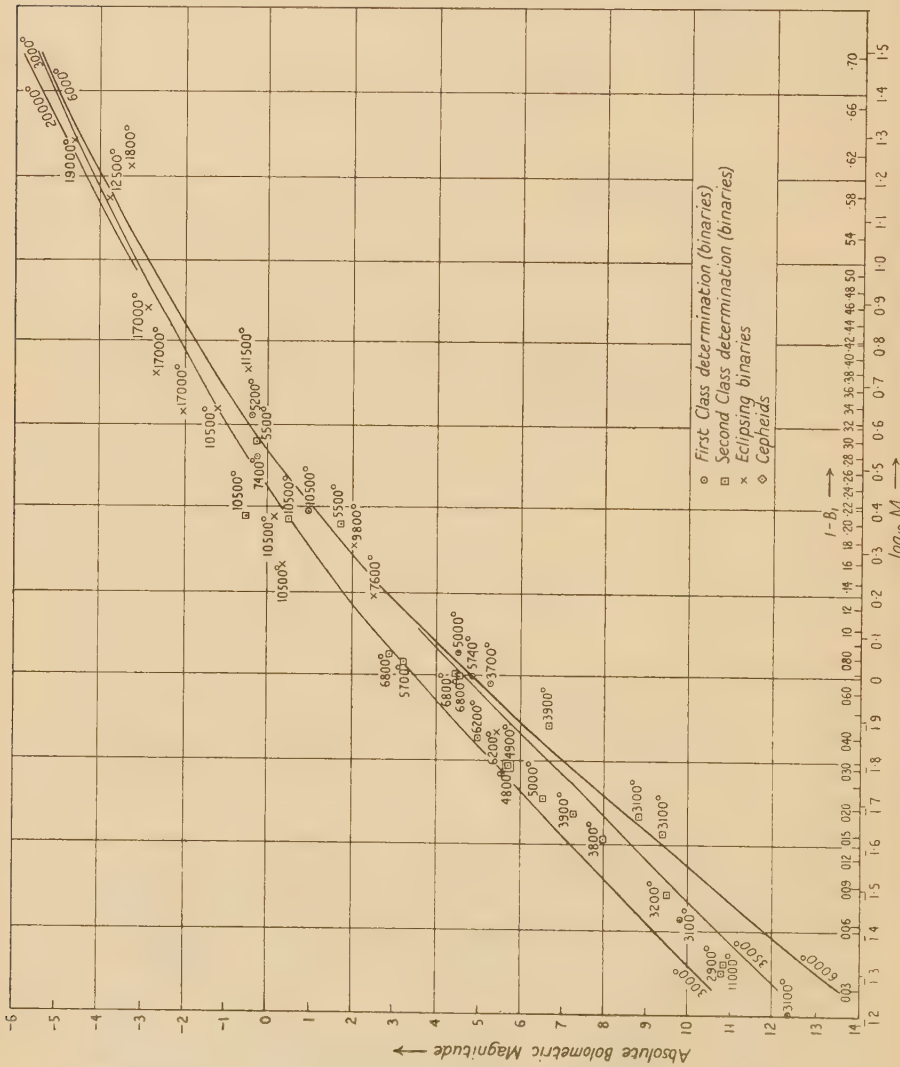
$$m = + 11{\cdot}85.$$

Eddington's computed value for Krüger 60*f* was 13·55.

We see that V-Puppis* has precisely the luminosity of V-Puppis; that Krüger 60**b* is $0^m{\cdot}29$ brighter than Krüger 60*b*; and that Krüger 60**f* is $0{\cdot}^m47$ brighter than Krüger 60*f*. Eddington's computations gave V-Puppis $0^m{\cdot}31$ too bright, Krüger 60*b* $1^m{\cdot}25$ too faint, and Krüger 60*f* $1^m{\cdot}23$ too faint.

I wish to make it quite clear that I am not attempting to predict luminosities of actual stars. We are simply comparing the observed luminosities of actual stars with the luminosities of the members of a certain homologous family. The result is astonishing. V-Puppis* and Krüger 60* have almost exactly the luminosities of V-Puppis and Krüger 60. In other words, our homologous family, chosen to reproduce Sun and Capella, also reproduces stars at such extreme parts of

the range as V-Puppis (high mass and luminosity) and Krüger 60 (low mass and luminosity). In between Sun* and Capella* we should rather have expected it to interpolate luminosities reproducing actual stars.




But we have no reason to expect it to reproduce luminosities outside this range.

26. We now turn to the stars as a whole. The luminosities of the actual stars given in Eddington's invaluable tables are plotted against the logarithms of their masses in fig. 1, with the effective temperature

written against each. The luminosities of our special homologous family are plotted † as curves on the same diagram, for temperatures 3000° and 6000°, with short curves for 3500° and 20,000° where these are appreciably different. It will be seen that the homologous family closely reproduces the observed luminosities throughout the range; it is difficult to detect any systematic deviation save that the 3000° curve is a little high at low masses. The scatter of the observations is of course considerable, and we cannot say that the homologous family reproduces any observed scatter due to temperature. What is, however, striking—and this is the important conclusion of our comparison with observation—is that the actual stars mimic a homologous family. In other words, as far as their observed luminosities, masses, and temperatures disclose any information about the internal density-distribution, that information is the same for all stars. C bears the appearance of a stellar constant. Conversely the single assumption that the stars have a common density-distribution, *i.e.* are describable by one and the same differential equation, is sufficient to give the observed luminosities of the stars for given mass and effective temperature, without any assumptions as to the physical state of the interior. Clearly, in view of our analysis, from the distribution of the actual luminosities in the mass-luminosity diagram we can by no means infer the gaseous nature of the interiors. A common density-distribution alone is indicated.

We conclude, then, that something in nature compels the stars to follow a common density-distribution. What this something is is a matter for future research. I have not of course extracted all the information that can be gleaned from the present investigation, for I have not discussed what can be learned from the numerical value of the coefficient C, now seen to be a stellar constant. Stability considerations may come in, as Jeans has suggested, and impose the common density-distribution.

27. The contents of Table IV., corrected according to Table V., are plotted in a different way in fig. 2. Here luminosity is plotted against temperature for constant mass. The observed stars are plotted on the same diagram. The diagram amounts to a Hertzsprung-Russell diagram turned right to left, with the mass-lines of a certain homologous family stamped upon it. The most striking feature of these mass-lines is the steep curve-up on the left-hand side. It is a consequence of the very low ionization in the photospheres of stars of low mass and low effective temperature; for such stars have high-pressure photospheres of low temperature, and since both low temperature and high pressure retard ionization, they give rise to low electron-pressure and so low opacity. Discussion of these curves is postponed to a later paper, but one remark may be made here.

Let us suppose something compels the star to adopt a particular density-distribution. Then for its actual mass and luminosity it must find a configuration of equilibrium in which its surface opacity is just appropriate to its mass, luminosity, and density-distribution. If its

† We plot the figures of Table IV., corrected according to Table V.

mass is low, ionization will be low in its photosphere owing to high pressure, and as it expands and its temperature falls, the ionization will decrease still further. This means that with expanding radius the photosphere becomes more transparent, ultimately very rapidly, and the star therefore cools at an increasing rate. When the rate of cooling

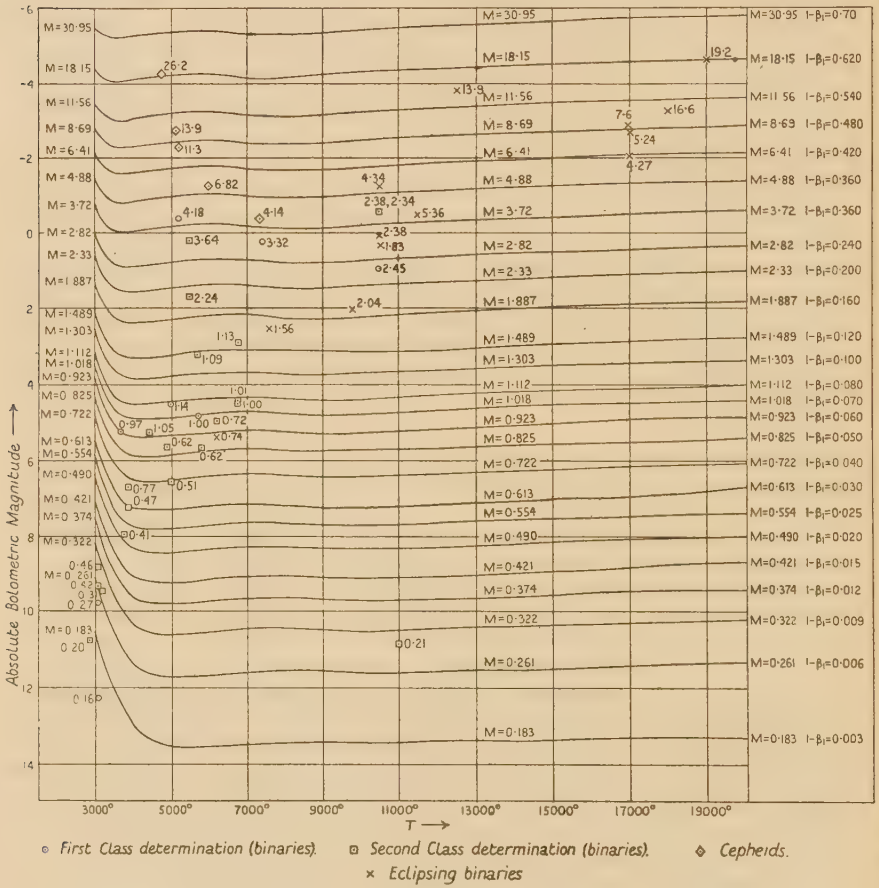



Fig. 2.

is equal to L the state will become steady. It is now clear that the star cannot expand too far at low temperatures without cooling at a rate exceeding the generation of energy. The sharp up-turns on the mass-curves of fig. 2 act as barriers preventing the star from taking a very low effective temperature. It is notable that this sharp bend occurs for dwarfs at between 3000° and 4000°. This seems to me to indicate why dwarf stars are not found at temperatures below 3000° ;

if they were put in a configuration in which the temperature was much lower than this they would cool so very rapidly that they would lose more heat than their energy supply provided. They would therefore contract and their surface temperatures would increase. 3000° or so is just the temperature at which under stellar pressures gases become ionized sufficiently to give appreciable opacity ; at lower temperatures the gases are so transparent at the star's boundary that energy leaks away too freely through the outer layers. The curves of fig. 2 show that the effect is less conspicuous for higher masses ; here the reduced pressure facilitates ionization, and so the opacity is less sensitive to decreasing temperature. We infer (always on the assumption that something compels the stars to have a common density-distribution) that giant stars can take lower temperatures than dwarf stars ; in other words, giant M or N stars can be redder than dwarf M's. I believe the facts are in general accordance with this prediction.

28. By way of conclusion I wish to distinguish between the general method of the paper and the particular results it arrives at. The general method is the part to which I attach importance. The result I have obtained by applying it—namely, that the stars are indistinguishable observationally (white dwarfs excepted) from a single homologous family with a common density-distribution—may be modified by an improved theory of photospheric opacity and ionization, or by better data for the luminosities, masses, and temperatures of the stars. It might then be discovered that C varied systematically over the Russell diagram—in other words, that there was a systematic change of density-distribution over the Russell diagram. The method, however, is independent of the special results found.

We have not sought to set up a mass-luminosity relation ; indeed, we have seen how equilibrium considerations can never of themselves determine anything so fundamental, since a given luminosity can correspond to any mass above a certain minimum. Nor have we sought to set up a mass-luminosity-temperature relation. Building up stars in mechanical equilibrium in all possible ways, we have classified these ways in homologous families. A member of a given homologous family is in neutral equilibrium as far as its mechanical equilibrium goes, and can take up any radius. But when its luminosity is given it adjusts its radius until the surface opacity is such that it cools at just the rate at which energy is being liberated. This provides us with a four-term relation connecting L, M, $T_1$ (or its equivalent, the radius or the photospheric opacity), and C, a coefficient depending on the density-distribution. The coefficient C, as it were, recapitulates in itself those aspects of the density distribution which are significant for the surface cooling ; it is a telescoped version of the structure of the star between centre and boundary. We can compute it from the form of the density-distribution ; we can only infer something about the density-distribution from the value of C. We can, however, compute C for any star of known L, M, and $T_1$ provided we know how to calculate the photospheric opacity. In this analysis no hypothesis is made as to the

nature of the interior of the star ; we assume only that the outer layers are gaseous and in radiative equilibrium.

The values of C for the actual stars seem to be about the same. We have ascertained this not directly, owing to uncertainty as to the true value of the photospheric opacity, but by constructing the homologous family (C constant), which contains two stars observationally indistinguishable from the Sun and Capella. This family is found to consist of stars whose luminosities, for given mass and temperature, coincide with the luminosities of the actual stars. Further, the photospheric opacity we find it necessary to attribute to our constructed homologous family is close to that obtained for the actual stars from spectroscopic data. Thus the constructed family could be made of real stellar material. The inference is that the actual stars (excluding white dwarfs) do in fact constitute a single homologous family, with a common density-distribution. This is an inference based solely on observation and not dependent on any theory of the interior.

If C is in fact a stellar constant, as observation appears to indicate, then the four-term relation reduces to a three-term relation between L, M, and $T_1$. The actual scatter of L, for given M and varying $T_1$, as given by this relation is small for temperatures of the observed stellar range, but the introduction of a genuine mass-luminosity law based on the energy-generating processes would be required to effect a calculation of stellar temperatures. If, however, we take over a genuine mass-luminosity law from observation of the type $L = L(M)$, then a glance at fig. 2 shows why stellar temperatures have their observed values.

A given horizontal line $L = L_1$ in this diagram meets any mass-curve in two points or in none. Thus, given L and M, there are either two configurations of equilibrium inside the homologous family or none. For large mass, the two configurations correspond to a B-star or a giant M-star ; it is tempting to suppose that where $L_1$ is so small that the horizontal line $L = L_1$ never intersects the curve $M = M_1$ and no configuration of equilibrium exists inside the homologous family, a white dwarf results. Thus we are led to the speculation that a large aggregate of matter M generating an arbitrary luminosity L will be born either as a B-star or as a giant M-star, or as a star of an entirely different structure. A scheme of stellar evolution reproducing the main features of the Hertzsprung-Russell diagram then results, but I leave the discussion of these speculations to a later paper.†

29. I should like to express my thanks to Professor Eddington for valuable criticisms of an earlier draft of this paper, though he is in no way responsible for the correctness or otherwise of the paper in its present

† The curves of fig. 2 show small humps in the region 6000°–7000°. These would give additional intersections with a horizontal line $L = L_1$ in certain cases, corresponding to the possible birth of a star as an F- or G-star. This is also in accordance with the Hertzsprung-Russell diagram. It would be highly interesting to have an accurate theory of photospheric opacity, so as to investigate the correct march of the curves of fig. 2 and the correctness of the conjectures here put forward concerning evolution.

form. It will be obvious to any reader how much the investigations of the present paper owe to Professor Eddington's writings. I should also like to express my indebtedness to pp. 895–902 of vol. ii. of Russell, Dugan, and Stewart's *Astronomy*; and to the various writings of H. Vogt. In particular, in *A.N.*, Bd. 233, Nr. 5569, p. 13 (footnote), Vogt points out that the fundamental equations connecting luminosity with the other characteristics of a star depend on the equation of state of a gas applied to the outer layers only. The paper also owes much to the writings of Sir James Jeans.

## Appendix.

### *The Cooling Sphere.*

The following simple example taken from elementary physics illustrates the point that in solving the problem of the flow of heat through a sphere we must not leave off the investigation at a point inside the boundary. I do not put forward this as a model of a star, because in the star mechanical equilibrium requires consideration. But it may be of interest to have a physical model in which the opacity distribution is important.

Let us imagine a sphere of copper, S, blackened at its surface and provided with a small electric heating coil near its centre. Let it cool freely in space, ultimately taking up a steady state. Now imagine a second sphere S′, also of copper save for the outer $\frac{1}{10}$ of its radius, which we suppose replaced by a layer of some badly conducting material like asbestos. Let its surface be blackened and a heating coil be provided at its centre as before. Let it cool freely in space. If the spheres are of equal radius and the heating coils are of equal strength, the surface temperatures of the two spheres will be equal, and the flow of heat at and just inside their boundaries will be equal. But the internal temperature distributions will be totally different. The sphere S′ coated with asbestos will be much hotter inside than the copper sphere S; for it is jacketed by a bad conductor. Now suppose the copper to have a very large coefficient of expansion. Then S′, being hotter, will expand more than S, and we shall have an observable effect, S′ being of larger radius than S. But we must avoid the error of arguing that since the flow of heat $\frac{1}{10}$ of the radius in from the boundary is the same in the two cases, therefore the internal distribution of temperature is the same in the two cases.

In this example it is the *distribution* of internal opacity (or conductivity) which is important. If S were provided with the same amount of asbestos as S′ but concentrated in a sphere at its centre, the copper in S would still be much colder than the equal amount of copper in S′, and so S′ would be larger than S in radius. Thus by observation of the radii, given the distribution of the energy-source, we could infer something of the distribution of conductivity.

The stellar problem is a kind of converse; if we knew the distribution of opacity we could infer something as to the distribution

of energy-liberation. It is shown in § 18 above how this can be carried out.

DESCRIPTION OF FIGURES.

Fig. 1.—Curves of absolute bolometric magnitude against mass ($\log_{10}$ M) for constant effective temperature, for a certain homologous family. The plotted points show the observed bolometric magnitudes of actual stars (taken from Eddington's lists, *Internal Constitution of the Stars*, pp. 154. 155), plotted against the logarithms of their observed masses, with the observed effective temperature written against each.

Fig. 2.—Curves of absolute bolometric magnitude against effective temperature (T) for constant mass M, for a certain homologous family. The plotted points show the observed bolometric magnitudes of actual stars plotted against their observed effective temperatures, with the mass written against each. (Data the same as in fig. 1.)

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## *Internal Circulation in Rotating Stars.*

By A. S. Eddington, F.R.S.

1. In 1924 H. von Zeipel * published an important theorem giving the relation

$$\epsilon \propto (1 - \omega^2/2\pi G\rho) \quad . \quad . \quad . \quad . \quad (1)$$

between the rate of generation of energy $\epsilon$ and the density $\rho$ in a star assumed to be rotating as a rigid body with angular velocity $\omega$. It was pointed out by H. Vogt † and the writer ‡ that this theorem gives no grounds for expecting that the relation (1) will be fulfilled in actual stars ; its non-fulfilment will lead to circulating currents in the interior. Since the conditions in a rotating star are not spherically symmetrical, the flow of radiation will in general tend to heat the polar and the equatorial regions unequally ; the unequal heating causes matter to flow in large-scale convection currents. Von Zeipel's condition determines a singular case in which there is no unequal heating, and the star is free from convection currents ; it can thus conform to the artificial restriction of rigid body rotation.

My purpose in this paper is to estimate the magnitude of the circulating currents that can be maintained by this unsymmetrical flow of radiation. Some results of interest follow from the mere existence of a circulation even if it be very slow. It produces a mixing which can scarcely fail to counteract the excessively slow diffusion of heavy elements to the centre and light elements to the outside of a star ; thus we are led to believe that the interior of a star has nearly uniform chemical composition throughout. The mixing also tends to equalise the angular momentum at different levels, and so to bring about an increase of angular velocity inwards—a condition also caused independently by radiative viscosity as shown by Jeans. But in other

* *Seeliger Festschrift*, p. 144, 1924. † *Astr. Nach.*, No. 5342, 1925.
‡ *Observatory*, **48**, 73, 1925.

applications an estimate of the speed of the currents is essential, especially in theories which connect solar phenomena with a large-scale internal circulation; we must not invoke in these theories a circulation of magnitude out of all proportion to the energy available to maintain it.

2. Internal currents affect both the mechanical and the thermal equilibrium of a star. I think that most stress has generally been laid on the disturbance of mechanical equilibrium, the idea being that the new forces (viscous, centrifugal, etc.) so modify the equations that, if von Zeipel's investigation were repeated with inclusion of these, the star would be found to have righted itself. But we have also to modify the conditions of thermal equilibrium by adding to the equations of radiative equilibrium terms expressing the movement of heat by convection. It will appear from the present discussion that this is by far the more important modification, and it may be studied independently of the mechanical complications.

Assume provisionally that the internal motions are too slight to introduce any appreciable new forces, so that the equations of mechanical equilibrium are the same as von Zeipel's. We have then only to amend his investigation to take account of heat convection. This can be done by noticing that in (1) $\epsilon$ represents the quantity of heat (per unit mass and time) that has unaccountably appeared in a region without our having traced its passage across the boundary, and is therefore presumed to have been generated internally. But if our watch on the frontier has been inadequate the heat appearing in this way will include not only "home products" but "smuggled goods." Accordingly we must now replace von Zeipel's $\epsilon$ by $\epsilon_1 + \epsilon_2$, where $\epsilon_1$ is the true rate of generation of energy and $\epsilon_2$ is the net gain of heat by convection.

But further, on the strength of the old meaning of $\epsilon$, von Zeipel has assumed that it is constant over a level surface. This now applies only to $\epsilon_1$; $\epsilon_2$ is necessarily positive at some points and negative at other points of a level surface, since any stream-line that crosses it must cross twice, once ascending and once descending. We have therefore to stop von Zeipel's investigation at the point where this assumption is about to be introduced, viz. at formula (198.7) in my *Internal Constitution of the Stars*:

$$-f(\phi)(-4\pi G\rho + 2\omega^2) - f'(\phi)(d\phi/dn)^2 = \rho(\epsilon_1 + \epsilon_2) \quad . \qquad (2)$$

Here $\phi$ is the potential (including centrifugal force), $d\phi/dn$ is the potential gradient along the normal to the level surface, and $f(\phi)$ is defined by (198.42),

$$H = -f(\phi)(d\phi/dn) \quad . \quad . \quad . \quad . \qquad (3)$$

where H is the outward flow of radiation. The only quantities in (2) not constant over a level surface are $\epsilon_2$ and $d\phi/dn$. Hence (over a level surface)

$$\text{range of } \epsilon_2 = \frac{f'(\phi)}{\rho} \times \text{range of } \left(\frac{d\phi}{dn}\right)^2 \quad . \quad . \qquad (3a)$$

Since there is as much upward as downward convection across a level surface, the values of $\epsilon_2$ will be about equally distributed between positive and negative, and the maximum value of $\epsilon_2$ will be about half the range.

The range of $(d\phi/dn)^2$, or, as it is usually called, $g^2$, can be estimated as a percentage of its mean value directly from the oblateness of the star. For example, a range of 30 per cent. on either side of the mean would evidently correspond to a highly deformed fast-rotating star. Let

$$\text{range of } g^2 = \pm q\overline{g^2} \quad . \quad . \quad . \quad . \quad (4)$$

Then by (3) we shall have for the extreme values of $\epsilon_2$

$$\epsilon_2 = \pm q\overline{g^2} f'(\phi)/\rho \quad . \quad . \quad . \quad . \quad (5)$$

With sufficient accuracy we may use the values for a spherical star in calculating the cofactor of $q$, so that

$$\epsilon_2 = \pm q \frac{f'(\phi)}{\rho}\left(\frac{d\phi}{dr}\right)^2 \quad . \quad . \quad . \quad . \quad (6)$$

Let $\mathrm{L}(= 4\pi r^2\mathrm{H})$ and M be the total generation of energy and the mass within the sphere of radius $r$. Then

$$\begin{aligned}\frac{f'(\phi)}{\rho}\left(\frac{d\phi}{dr}\right)^2 &= \frac{g}{\rho}\frac{d}{dr}f(\phi) = -\frac{g}{\rho}\frac{d}{dr}\left(\frac{\mathrm{H}}{g}\right) \text{ by (3)} \\ &= -\frac{\mathrm{GM}}{r^2\rho}\frac{d}{dr}\left(\frac{\mathrm{L}}{4\pi\mathrm{GM}}\right) \\ &= -\mathrm{M}\frac{d}{d\mathrm{M}}\left(\frac{\mathrm{L}}{\mathrm{M}}\right) \quad \text{since} \quad d\mathrm{M} = 4\pi\rho r^2 dr \\ &= \frac{\mathrm{L}}{\mathrm{M}} - \frac{d\mathrm{L}}{d\mathrm{M}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)\end{aligned}$$

Now L/M is the mean rate of generation of energy interior to $r$, denoted by $\bar{\epsilon}_1$; and $d\mathrm{L}/d\mathrm{M}$ is the rate of generation of energy at $r$ denoted by $\epsilon_1$. Hence (6) becomes

$$\epsilon_2 = \pm q(\bar{\epsilon}_1 - \epsilon_1) \quad . \quad . \quad . \quad . \quad (8)$$

An upper limit to $\epsilon_2$ is thus $\pm q\bar{\epsilon}_1$. This will be approximately attained in the outer parts of the star if (as is generally supposed) the source of stellar energy is mainly situated in the deep interior. In the Sun $\bar{\epsilon}_1 = 1{\cdot}9$ ergs per gram per sec.; $q$ is small since the Sun's rotation is slow. Taking $q = 0{\cdot}1$, so that $\epsilon_2$ ranges between $+ 0{\cdot}2$ and $- 0{\cdot}2$ ergs/gm. sec., we have a fairly safe upper limit to the heat supplied by convection.

3. If a gas moves adiabatically to a place of slightly different pressure, its change of temperature $\delta\mathrm{T}$ is given by

$$\frac{\delta\mathrm{T}}{\mathrm{T}} = \frac{\gamma - 1}{\gamma}\frac{dp}{p}.$$

If the surrounding temperature in the new position is $T + dT$, the usual theory of stellar equilibrium gives (approximately)

$$\frac{dT}{T} = \frac{1}{4}\frac{dp}{p}.$$

Thus the convection current brings into a unit volume additional heat equal to

$$C_v\rho(\delta T - dT) = C_v\rho T\left(\frac{\gamma - 1}{\gamma} - \frac{1}{4}\right)\frac{dp}{p}.$$

Or since $p = (\gamma - 1)C_v\rho T$, the additional heat (per unit volume) is

$$\left(\frac{1}{\gamma} - \frac{1}{4(\gamma - 1)}\right)dp = \alpha dp, \text{ say} \quad . \quad . \quad . \quad (9)$$

The maximum value of $\alpha$ is when $\gamma$ has its maximum value $\frac{5}{3}$; then $\alpha = \frac{9}{40}$. For $\gamma = 1{\cdot}4$, $\alpha = \frac{1}{11}$ approximately. $\alpha$ becomes zero for $\gamma = \frac{4}{3}$, but it is unlikely that we have to do with very low values, since not only is the value of $\gamma$ obtained by direct calculation for material in stellar conditions opposed, but also if $\gamma$ approaches $\frac{4}{3}$ the star is on the verge of instability. We may take $\alpha = 0{\cdot}04$ as a reasonably safe lower limit.

4. We can now equate the heat convection calculated in §§ 2 and 3, introducing a factor $\rho$ because (8) refers to unit mass and (9) to unit volume. Thus

$$\alpha dp = \pm q\bar{\epsilon}_1\rho \quad . \quad . \quad . \quad . \quad (10)$$

This determines the change of pressure per second of gas travelling with the convection current. Since $dp = -g\rho dr$, and $dr$ (the change of $r$ per second) is the vertical velocity $v$ of the current, we have

$$v_r = \pm\frac{q\bar{\epsilon}_1}{\alpha g} \quad . \quad . \quad . \quad . \quad (11)$$

For the sun $g = 2{\cdot}7.10^4$ at the surface and increases downwards. With the values of the other constants already given, the result is

$$v_r < 2{\cdot}10^{-4} \text{ cm. per sec., or 60 metres per year.}$$

Evidently the circulation maintained by the unsymmetrical flow of radiation is on a far smaller scale than that postulated in Bjerknes's theory of the sunspot period, etc., and recently by Rosseland * in connection with variable stars.

Since the velocities are so small, our provisional assumption that the additional mechanical forces are insignificant is fully justified. It is true that we have calculated only the maximum *vertical* velocity of the current; the horizontal velocities in a meridian plane will presumably be somewhat larger, but it is easily seen that the equation of continuity forbids any great disparity in magnitude.

* Avhandlinger Norske Videnskaps-Akademi, 1929, No. 6.

We set no upper limit to currents in the third direction, *i.e.* circulating round the axis of rotation ; or, as it would more usually be expressed, we do not restrict the variation of $\omega$ in any manner through the star. Variation of $\omega$ is best treated as an independent question, even though (as already suggested) an increase of $\omega$ inwards may be one of the remote consequences of the existence of currents in a meridian plane. Of course, the numerical results for a star of varying $\omega$ will not be quite the same as our results for a star of uniform $\omega$, but there is no reason to expect any difference in the order of magnitude. In particular there is no warrant for the idea that the variation of $\omega$ in actual stars tends to be such as to eliminate the unequal heating ; * that speculation, I think, arose from the idea of von Zeipel's condition as one which an actual star would try to conform to, instead of as one which would simplify matters for the mathematician if the star happened to conform to it.

It may be of interest to note the direction of the circulation. Since $f'(\phi)$ is positive by (7), and $d\phi/dn$ is a minimum at the equator, we see from (2) that $\epsilon_2$ is positive at the equator. This means that the current is downwards along an equatorial radius, as may be verified from (9). (In a stable atmosphere a downward current is continually being compressed to a temperature higher than its surroundings, and so warms the regions it passes through.) Thus the circulation near the surface is from the poles to the equator.

5. *Summary.*—The unsymmetrical flow of radiation in a rotating star, by its unequal heating effect, maintains currents circulating in the meridian planes ; but such currents are limited by the condition that the vertical velocity does not exceed

$$v_r = q\bar{\epsilon}_1/\alpha g,$$

where $\bar{\epsilon}_1$ is the mean rate of generation of energy (per gm. per sec.) interior to the point considered ;

$q$ is a function of the oblateness having the definition—maximum deviation of $g^2$ on the level surface from its mean value $= q$ times the mean value.

$\alpha$ is a factor depending on the adiabatic constant $\gamma$ of the material, with a maximum value of about $\frac{1}{4}$, and approaching zero as the star approaches instability. It may well be as low as $\frac{1}{2}$ or $\frac{1}{4}$ of this maximum ; but to attempt to obtain a large $v_r$ by taking $\alpha$ nearly zero would (besides postulating a very improbable coincidence of $\gamma$ with a critical value) merely amount to saying that in a medium on the verge of instability very large motions can arise from very small causes.

For the Sun $v_r$ is about 60 metres per year. The circulation is therefore much slower than has sometimes been supposed.

* *Cf.* Gerasimovič, *Observatory*, **48**, 148, 1925.

## *The Effect of Distribution of Density on the Period of Pulsation of a Star.* By J. C. P. Miller, B.A.

(*Communicated by Professor A. S. Eddington.*)

1. The period of adiabatic pulsation of a gaseous star has been investigated by A. S. Eddington.* His calculations relate to a model star corresponding to the case $n = 3$ in Emden's equation:

$$\frac{d^2u}{dz^2} + \frac{2}{z}\frac{du}{dz} + u^n = 0.$$

At Professor Eddington's suggestion the calculations are here extended to the models $n = 2$ and $n = 4$. For $n = 2, 3, 4$, the central densities are respectively 11·4, 54·4, 623 times the mean density, so that a comparison of the results for the three models will show how the period is affected by the degree of concentration of the mass towards the centre.

In the first investigations we have taken $\gamma$, the effective ratio of specific heats for the matter and radiation combined, to be constant throughout the star (as it is in Eddington's model $n = 3$). We then find (for stars having the same mass, radius, and $\gamma$) that the model $n = 2$ gives a longer period than $n = 3$, and the model $n = 4$ gives a shorter period; that is to say, the greater the concentration towards the centre the shorter the period.

Actually when $n \neq 3$ the value of $\gamma$ varies between the centre and the outside of a star, and we have therefore made a second investigation, which takes account of this variation, for the case $n = 2$. For this calculation the model considered by E. A. Milne † is appropriate, in which the alteration of the index $n$ is brought about by taking $1 - \beta$ (the ratio of radiation-pressure to total pressure) to be a function of $u$ given by $1 - \beta \propto u^{3-n}\beta^4$. Then since $\gamma$ is a function of $1 - \beta$ and $\Gamma$ (the ratio of specific heats of the matter alone), its value can be found at all points. The results are not appreciably different from those of the first approximation, provided that the varying $\gamma$ is replaced by an appropriate average value.

Eddington finds it necessary to adopt a low value of $\Gamma$, viz. 1·40, in order to make his model fit the observed periods of the Cepheid variables.‡ The object of this paper is to show that by changing the index $n$, the observations are fitted by a value of $\Gamma$ nearer to $\frac{5}{3}$, which is on other grounds considered to be likely for matter in stellar conditions. This higher value is obtained by reducing $n$; accordingly, we may perhaps take this paper as evidence that, at any rate in the Cepheid stage, actual stars are less concentrated towards the centre than the model $n = 3$.

On the other hand, there is the possibility (suggested by Eddington) that the occurrence of pulsation may be attributable to an exceptionally

* *M.N.*, **79**, 2, 1918; *Internal Constitution of the Stars*, p. 186.
† *M.N.*, **87**, 708, 1927.
‡ *Internal Constitution of the Stars* (hereafter referred to as *I.C.S.*), p. 193.

low value of $\Gamma$ ; in that case we should not be justified in altering $n$ so as to correspond to the value of $\Gamma$ considered likely for stars in general.

2. Following Eddington's method (*I.C.S.*, § 127), we take $P \propto \rho^\gamma$, with $\gamma$ constant for a particular piece of matter but varying throughout the star. Then $P_1 = \gamma\rho_1$, and we reach his equations (127.51) and (127.52), which (with some change of notation, using Emden's units) may be written

$$P_0\frac{dP_1}{dz} - g_0\rho_0 P_1 = \rho_0(4g_0 + \nu^2 z)y \quad . \quad . \quad . \quad (2.1)$$

$$P_1 = -\gamma\left(3y + z\frac{dy}{dz}\right) \quad . \quad . \quad . \quad (2.2)$$

our $y$ being Eddington's $\xi_1$. Eliminating $P_1$, we have

$$\frac{d^2y}{dz^2} + \left(\frac{4-\mu}{z} + \frac{1}{\gamma}\frac{d\gamma}{dz}\right)\frac{dy}{dz} + \left\{\frac{\nu^2\rho_0}{\gamma P_0} - \left(3 - \frac{4}{\gamma}\right)\frac{\mu}{z^2} + \frac{3}{\gamma z}\frac{d\gamma}{dz}\right\}y = 0 \quad (2.3)$$

where $\mu = g_0\rho_0 z/P_0$. We find from Emden's theory,

$$\mu = -(n+1)\frac{z}{u}\frac{du}{dz},$$

$$\frac{\rho_0}{P_0} = \frac{1}{u}\left(\frac{\rho_0}{P_0}\right)_c,$$

where the suffix $c$ refers to values at the centre of the star. Put

$$p^2 = \frac{\nu^2}{\gamma_c}\left(\frac{\rho_0}{P_0}\right)_c,$$

$$\alpha = 3 - 4/\gamma.$$

Then (2.3) becomes

$$\frac{d^2y}{dz^2} + \left(\frac{4-\mu}{z} + \frac{1}{3-\alpha}\frac{d\alpha}{dz}\right)\frac{dy}{dz} + \left(\frac{p^2}{u}\frac{3-\alpha}{3-\alpha_c} - \frac{\alpha\mu}{z^2} + \frac{3}{3-\alpha}\frac{1}{z}\frac{d\alpha}{dz}\right)y = 0 \quad (2.4)$$

We have to solve this equation numerically.

In the actual calculations we have in all cases adopted $\alpha_c = 0{\cdot}2$. We have then to determine (by trial and error) the value of $p^2$, which will give the first node at the boundary of the star. Higher harmonics are not considered. We start with an arbitrary amplitude $y = 1$ at the centre ; by symmetry $y'(\equiv dy/dz)$ vanishes there ; and we proceed outwards step by step by quadratures. In the interior of a star the condition for a node is $P_1 = -\gamma(3y + zy') = 0$, but at the boundary it is sufficient that $y$ and $y'$ remain finite as $P_0$ vanishes, so that $P_0P_1$ may vanish. We find in fact that, if $p_0^2$ gives the node at the boundary, for values of $p^2$ near $p_0^2$ the value of $y$ tends to $+\infty$ or $-\infty$ at the boundary according as $p^2 <$ or $> p_0^2$.

3. First take $\gamma$ constant so that the equation (2.4) becomes the same as Eddington's (127.8),

$$\frac{d^2y}{dz^2} + \frac{4-\mu}{z}\frac{dy}{dz} + \left(\frac{p^2}{u} - \frac{\alpha\mu}{z^2}\right)y = 0.$$

The results of our numerical calculation (for $\alpha = 0{\cdot}2$) are

$$n = 2 \qquad p^2 = {\cdot}074$$
$$n = 4 \qquad p^2 = {\cdot}02$$

We can add for comparison the case of uniform density $n = 0$, which admits of elementary solution. For then

$$u = 1 - \frac{1}{6}z^2, \qquad \mu = -\frac{z}{u}\frac{du}{dz} = \frac{z^2}{3u},$$

and the solution finite at the boundary is obtained by taking $p^2/u - \alpha/3u = 0$, or

$$p^2 = \tfrac{1}{3}\alpha.$$

Again, for $n = 3$ Eddington has found

| | | | | |
|---|---|---|---|---|
| $\alpha =$ | 0 | 0·1 | 0·2 | 0·6 |
| $p^2 =$ | 0 | 0·0315 | 0·060 | 0·156 |

Collecting the results for $\alpha = 0{\cdot}2$ we have

| | | | | |
|---|---|---|---|---|
| $n =$ | 0 | 2 | 3 | 4 |
| $p^2 =$ | ·067 | ·074 | ·060 | ·02 |

4. As in Eddington, § 130, we have, whether $\gamma$ is constant or variable,

$$p^2 = \frac{\nu^2}{\gamma_c}\left(\frac{\rho_0}{\mathrm{P}_0}\right)_c\left(\frac{\mathrm{R}}{\mathrm{R}'}\right)^2 \quad . \quad . \quad . \quad . \quad (4.1)$$

R being the radius of the star in ordinary units and R′ in Emden's unit. By Eddington (55.41), (55.42),

$$\frac{{\rho_c}^2}{\mathrm{P}_c} = \frac{{\phi_c}^{n-1}}{(n+1)^{n-1}\kappa^n},$$

and by Eddington (57.3),

$${\phi_c}^{n-1} = \left(\frac{\mathrm{GM}}{\mathrm{M}'}\frac{\mathrm{R}'}{\mathrm{R}}\right)^{n-1}$$

$$[(n+1)\kappa]^n = 4\pi\mathrm{G}\left(\frac{\mathrm{GM}}{\mathrm{M}'}\right)^{n-1}\left(\frac{\mathrm{R}'}{\mathrm{R}}\right)^{n-3}.$$

Hence

$$\frac{{\rho_c}^2}{\mathrm{P}_c}\left(\frac{\mathrm{R}}{\mathrm{R}'}\right)^2 = \frac{n+1}{4\pi\mathrm{G}} \quad . \quad . \quad . \quad . \quad (4.2)$$

From (4.1) and (4.2),

$$p^2 = \frac{\nu^2}{\gamma_c\rho_c}\frac{n+1}{4\pi\mathrm{G}} \quad . \quad . \quad . \quad . \quad (4.3)$$

The period $\Pi$ is $2\pi/\nu$. Hence

$$\Pi\sqrt{\rho_c} = \sqrt{\frac{(n+1)\pi}{\mathrm{G}}} \cdot (\gamma_c p^2)^{-\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (4.4)$$

Or, since we wish to compare the periods of different models with the same *mean density* $\rho_m$,

$$\Pi\sqrt{\rho_m} = \sqrt{\frac{(n+1)\pi}{G}} \cdot (\gamma_c p^2 \rho_c/\rho_m)^{-\frac{1}{2}} \quad . \quad . \quad (4.5)$$

If for the moment we neglect the change of $p$ with $n$, the period for different models is proportional to $\{(n+1)\rho_m/\rho_c\}^{\frac{1}{2}}$, or to 1, ·517, ·271, ·090, for $n = 0, 2, 3, 4$. These figures must then be divided by the corresponding values of $p$ determined from the quadratures in § 3.

5. When $n = 0$, $p^2 = \frac{1}{3}\alpha$; also the values already given for $n = 3$ show that in that case $p^2$ can be represented rather roughly as proportional to $\alpha$. Although we have not directly tested this proportionality for $n = 2$ and $n = 4$, we shall assume that it holds approximately, so that the ratio of the periods in different models is the same whatever the value of $\alpha$. If $\Pi_2, \Pi_3, \Pi_4$ are the periods for $n = 2, 3, 4$ respectively, we find

$$\Pi_2/\Pi_3 = 1{\cdot}703 \qquad \Pi_3/\Pi_4 = 1{\cdot}751 \quad . \quad . \quad (5.1)$$

these ratios being determined from our calculations for $\alpha = 0{\cdot}2$, but now assumed to be approximately true generally. It may be pointed out in justification of this approximation that the main part of the change of period is contributed by the term $\{(n+1)\rho_m/\rho_c\}^{\frac{1}{2}}$, the value of $p$ being almost steady between $n = 0$ and $n = 3$.

The above ratios suggest that we should use an empirical relation,

$$\Pi_n/\Pi_3 = C^{3-n} \qquad (2 < n < 3) \quad . \quad . \quad . \quad (5.2)$$

where C will be taken as 1·703, since the range $n = 2$ to $n = 3$ is the most important for us. We shall also use the same relation and value of C for the range $(1 < n < 2)$.

We can now apply our results to δ Cephei, determining the value of $n$, which fits the supposed value $\Gamma = \frac{5}{3}$, for the ratio of specific heats of stellar material. Eddington found with $\Gamma = \frac{5}{3}$ and $n = 3$ a theoretical period $\Pi_3 = 3{\cdot}15$ days. By (5.2) we deduce $\Pi_2 = 5{\cdot}36$ days, which is precisely the observed period. Hence if we are right in believing that $\Gamma$ is not much below its limiting value $\frac{5}{3}$, the concentration of mass in δ Cephei is not much greater than that given by the model $n = 2$.

6. We now consider the variation of $\gamma$ with $z$, when $n = 2$. From Milne, *loc. cit.*, equation (4), with $s = 1$,

$$(1-\beta)/\beta^4 = u(1-\beta_c)/\beta_c{}^4 \quad . \quad . \quad . \quad (6.1)$$

And from Eddington (129.6) we find

$$\frac{1}{\alpha} = \left(1 + \frac{1}{3 - 4/\Gamma}\right)\frac{1}{\beta} - \frac{\beta}{4 - 3\beta} \quad . \quad . \quad . \quad (6.2)$$

From these we calculate the variation of $\alpha$ throughout the star for any assumed value of $\alpha_c$ and $\Gamma$.

We again adopt $\alpha_c = 0{\cdot}2$ and try $\Gamma = \frac{5}{3}$. By (6.2) $\beta_c = {\cdot}51205$,

and the resulting mass of the star is 8·365 ⊙, by Milne, equation (16). The numerical solution now gives

$$p^2 = \cdot 0837.$$

The most illuminating way of expressing this result is to state the equivalent constant value of $\alpha$ which could replace the varying $\alpha$. For $n = 2$, with constant $\alpha$, we found $p^2 = \cdot 074$ for $\alpha = 0{\cdot}2$ ; assuming as before that $p^2$ varies approximately as $\alpha$, this gives $p^2 = \cdot 037\alpha$. The above determination of $p^2$ is accordingly found to correspond to an equivalent average value

$$\bar{\alpha} = \cdot 226.$$

It is useful to compare this with an average formed in a simpler way. An appropriate average value of $\beta$ is the constant value $\bar{\beta}$ determined for a star of the same mass constructed according to Eddington's model $n = 3$. From $\Gamma$ and $\bar{\beta}$ we deduce a corresponding $\bar{\alpha}$ by (6.2). The result is

$$\bar{\alpha} = \cdot 223.$$

The agreement is very close, and we shall therefore assume that it is always permissible to calculate the equivalent $\bar{\alpha}$ by this second simpler method. This enables us to avoid repeating a long investigation by quadratures for each star separately.

The consequent correction to the results for $\delta$ Cephei already obtained in § 5 is very slight. We there compared $\Pi_2$ and $\Pi_3$ for the same value of $\gamma$ or $\alpha$, and since we used Eddington's calculations for the model $n = 3$, we took over his calculation of $\alpha$ for that model. This is $\bar{\alpha}$ according to the second of the above definitions, and as shown above it is permissible to use it in calculating $p^2$. But theoretically a small correction is still needed, because $\gamma_c$ in (4.5) is not the same as Eddington's $\bar{\gamma}$. Corresponding to

$$\alpha_c = 0{\cdot}2 \qquad \bar{\alpha} = \cdot 223$$

we have

$$\gamma_c = 1{\cdot}429 \qquad \bar{\gamma} = 1{\cdot}440,$$

so that

$$(\gamma_c/\bar{\gamma})^{-\frac{1}{2}} = 1{\cdot}0038.$$

As the correction is smaller than quantities already ignored it is unnecessary, and perhaps misleading, to apply it.

7. Combining (5.2) with (4.5) together with the approximate relation $p^2 \propto \alpha$, our results may be summarised by the formula

$$\Pi\sqrt{\rho_m} = \mathrm{A}C^{3-n}(\gamma_c\bar{\alpha})^{-\frac{1}{2}} \quad . \quad . \quad . \quad (7.1)$$

where

$$C^2 = [(n+1)\rho_m\bar{\alpha}/\rho_c p^2]_{n=2} \div [(n+1)\rho_m\bar{\alpha}/\rho_c p^2]_{n=3}.$$

Or numerically

$$\Pi\sqrt{\rho_m} = \cdot 03936(1{\cdot}703)^{3-n}(\gamma_c\bar{\alpha})^{-\frac{1}{2}} \text{ days} \quad . \quad . \quad (7.2)$$

It saves a great deal of probably unnecessary calculation to replace $\gamma_c$ in this formula by $\bar{\gamma}$, since the consequent change of period is never more than 1 per cent., and as in the example above given is less than the error involved in our other approximations. Then $\bar{\gamma}\bar{a}$ is the same as for the model $n = 3$.

We apply (7.2) to the Cepheids given by Eddington in *Internal Constitution of the Stars*, Tables XXIV. and XXV., and determine the value of $n$ which agrees with the observed periods for $\Gamma = \frac{5}{3}$. The mean results are:

For stars Nos. 5– 9 . . $n = 2{\cdot}164$
,, ,, ,, 10–14 . . $n = 1{\cdot}815$

and for all ten stars $n = 1{\cdot}990$. These are the groups chosen by Eddington in obtaining his mean value $\Gamma = 1{\cdot}40$ for $n = 3$.

For these ten stars $n$ ranges between 1·47 and 2·34. Among the eighteen stars listed by Eddington there is only one ($l$ Carinæ) which gives a value greater than 3, viz. 3·20.

8. Emden's tables being insufficiently complete for the present purpose, the values of $u$, etc., have been recalculated for $n = 2$ and $n = 4$. The following are the results for $n = 2$. The table also includes the auxiliary quantity $\mu$, and the values of $\beta$ and $\alpha$ for the special conditions $\Gamma = \frac{5}{3}$, $a_c = 0{\cdot}2$.

| $z$. | $n$. | $-u'$. | $\mu$. | $\beta$. | $\alpha$. |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | ·51205 | ·2 |
| ·25 | ·98965 | ·08230 | ·06237 | ·51311 | ·20046 |
| ·5 | ·95935 | ·15860 | ·24798 | ·51627 | ·20183 |
| ·75 | ·91128 | ·22383 | ·55266 | ·52152 | ·20412 |
| 1·0 | ·84865 | ·27454 | ·97050 | ·52885 | ·20733 |
| 1·25 | ·77532 | ·30920 | 1·4955 | ·53822 | ·21146 |
| 1·5 | ·69532 | ·32811 | 2·1235 | ·54962 | ·21654 |
| 1·75 | ·61239 | ·33295 | 2·8544 | ·56307 | ·22261 |
| 2 | ·52976 | ·32629 | 3·6955 | ·57859 | ·22971 |
| 2·25 | ·44993 | ·31109 | 4·6670 | ·59628 | ·23795 |
| 2·5 | ·37467 | ·29016 | 5·8083 | ·61631 | ·24751 |
| 2·75 | ·30511 | ·26598 | 7·1920 | ·63898 | ·25863 |
| 3 | ·24178 | ·24057 | 8·9551 | ·66480 | ·27176 |
| 3·25 | ·18479 | ·21544 | 11·367 | ·69463 | ·28765 |
| 3·5 | ·13395 | ·19164 | 15·022 | ·73000 | ·30773 |
| 3·75 | ·08881 | ·16982 | 21·512 | ·77389 | ·33504 |
| 4 | ·04885 | ·15038 | 36·944 | ·83304 | ·37792 |
| 4·25 | ·01342 | ·13342 | 126·72 | ·92902 | ·47974 |
| 4·35305 | 0 | ·12721 | ∞ | 1 | ·6 |

I am greatly indebted to Professor Eddington for his valuable suggestions and help.

## *Irregularities of Period of Long-period Variable Stars.*

By A. S. Eddington and S. Plakidis.

1. The light fluctuations of long-period variables have a well-marked periodicity complicated by superposed irregularities. There are various possible types of irregularity, and the question arises whether we can infer from the accumulated observational data which types are operative. The following types of irregularity suggest themselves for consideration :—

(1) The individual period (*i.e.* interval between two successive maxima) may differ from the mean period by a purely accidental fluctuation independent of previous fluctuations.

(2) The individual period may be correlated to those preceding it, either

(*a*) positively, so that periods longer than the average tend to occur in groups, or

(*b*) negatively, so that a period longer than the average is likely to be succeeded by a period shorter than the average.

(3) The irregularity may be a fluctuation of phase rather than of period ; that is to say, the primary cause of the light change may have an entirely regular period, but its visible effect may be delayed or accelerated by casual circumstances, so that the date of maximum differs from the ephemeris date by an accidental fluctuation.

(4) The casual " circumstances " in (3) may be long enduring so as to affect a succession of maxima ; the delay of one maximum is then correlated positively to the delay of the preceding maximum.

(5) The irregularity may be due to periodic causes.

The primary aim of the present investigation is to test whether the observed times of maxima are consistent with the hypothesis that the irregularities are of type (1). The stars examined are *o* Ceti and $\chi$ Cygni, for which long-continued observations are available. The feature of type (1) irregularity is that the delays and accelerations of successive periods mount up in the same way as accidental errors. If the star is late, the time lost is written off as irretrievable. There is no tendency to return to the original ephemeris dates, but in the ensuing periods gains or further losses occur as chance may determine.

2. Having determined a mean period from a very long series of observations, we construct a uniform ephemeris and compute the differences of date (observed—calculated) for each observed maximum. Let $a_r$ be the number of days late of the $r$th maximum compared with the ephemeris, and let

$$u_x(r) = a_{r+x} - a_r \quad . \quad . \quad . \quad . \quad (1)$$

Then $u_x(r)$ is the accumulated delay in $x$ periods which according to hypothesis (1) is the sum of $x$ uncorrelated accidental fluctuations. By the ordinary theory of errors, the probable value of the sum of $x$ fluctuations is $\sqrt{x}$ times the probable value of one fluctuation.

If $\bar{u}_x$ is the average value (without regard to sign) of $u_x(r)$ for as many values of $r$ as the observational material admits, we should have accordingly

$$\bar{u}_x = \epsilon\sqrt{x} \quad . \quad . \quad . \quad . \quad (2)$$

where $\epsilon = \bar{u}_1$.

By way of contrast, consider hypothesis (3). In this case $a_r$ is a purely accidental deviation, and its probable value is independent of $r$. If then the average value of $| a_r |$ for all values of $r$ is $\alpha$, the average value of $| a_{r+x} - a_r |$ is $\alpha\sqrt{2}$. Hence

$$\bar{u}_x = \alpha\sqrt{2} \quad . \quad . \quad . \quad . \quad (3)$$

It is easily seen that if fluctuations of both kinds occur, the result will be

$$\bar{u}_x = \sqrt{(2\alpha^2 + x\epsilon^2)} \quad . \quad . \quad . \quad . \quad (4)$$

We must in any case expect that the observational data will give an $\alpha$ term, because accidental error in determining the date of maximum is equivalent to accidental fluctuations of type (3). We shall accordingly examine whether the observations satisfy a formula of the form (4).

A correcting factor is required when $x$ is a considerable fraction of the whole number of periods employed in determining the mean period. Let $\epsilon_1, \epsilon_2 \ldots \epsilon_n$ be the accidental fluctuations of successive periods from the true mean period, then the error of the deduced mean period is $(\epsilon_1 + \epsilon_2 + \ldots + \epsilon_n)/n$. The deviation of the first period from the *adopted* mean period is therefore

$$\epsilon_1 - (\epsilon_1 + \epsilon_2 + \ldots + \epsilon_n)/n.$$

Forming similar expressions for the second, third, . . . periods, and adding the first $x$ of them, we have

$$u_x(0) = \frac{n-x}{n}(\epsilon_1 + \epsilon_2 + \ldots + \epsilon_x) - \frac{x}{n}(\epsilon_{x+1} + \epsilon_{x+2} + \ldots + \epsilon_n).$$

Whence by the theory of combination of independent deviations

$$\left.\begin{aligned} \bar{u}_x &= \epsilon\sqrt{\left\{\left(\frac{n-x}{n}\right)^2 x + \left(\frac{x}{n}\right)^2 (n-x)\right\}} \\ &= \epsilon\sqrt{x} \cdot \sqrt{(1 - x/n)} \end{aligned}\right\} \quad . \quad . \quad (5)$$

This replaces (2); but in our applications $x$ is small compared with $n$, and it has not been necessary to use the correcting factor.

3. The observed dates of maximum for $o$ Ceti and $\chi$ Cygni have been taken from *Harvard Annals*, **55**, supplemented up to the year 1927 inclusive from the *Reports of the V.S.S. of the B.A.A.** Wherever in the former source more than one date for the same maximum is given, the mean has been adopted. From these data mean periods of 331·4

* These last data were kindly supplied by Mr. F. de Roy, Director of the V.S.S.

days for *o* Ceti and 406·5 days for $\chi$ Cygni were deduced, and uniform ephemerides were computed with the formulæ

$$M = 2402985 + 331{\cdot}4\,E$$
$$M = 2389524 + 406{\cdot}5\,E$$

respectively.

Former discussions of these stars have been made by T. E. R. Phillips and W. J. Luyten. The former (*J.B.A.A.*, **27**, No. 1, 1916), giving the date curve of maxima, remarks that "both show very large irregularities, and it looks rather as if in these stars the time of maximum is affected by continuous but non-periodic change." Dr. Luyten, in his *Observations of Variable Stars* (Leiden, 1921), giving similar diagrams of observed—computed maximum, is "inclined to think that the period of *o* Ceti changes continuously and not abruptly as Professor Turner suggests, although the latter finds remarkable regularities in these changes." As regards $\chi$ Cygni, in discussing the results obtained by Professor Turner (*M.N.*, **80**, 282 and 481, 1920), he says that the period of this star being fairly constant in the beginning, from a certain point becomes markedly larger, and at last shows a tendency to decrease.

There are many breaks in the earlier series of observations, certain maxima being unobserved, so that it is a matter of accident whether a particular $u$, say $u_{10}(25)$, is known and included in $\bar{u}_{10}$ or left out as unknown. This should make no difference if the whole number of observations is very great; but in a limited series it has probably had the effect of making the curves less smooth than they would have been.

4. The curves in fig. 1 show $\bar{u}_x$ as ordinate plotted against $x$ as abscissa. The jagged curves are calculated from the observations of *o* Ceti (I.) and $\chi$ Cygni (II.). The smooth curves represent the best fit * with the theoretical formula $\bar{u}_x = \sqrt{(2\alpha^2 + x\epsilon^2)}$, the constants being

| | | |
|---|---|---|
| For *o* Ceti | $\alpha = 5{\cdot}48$ | $\epsilon = 4{\cdot}48$ |
| „ $\chi$ Cygni | $\alpha = 1{\cdot}58$ | $\epsilon = 5{\cdot}66$ |

The unit is 1 day.

Although each value of $\bar{u}_x$ is the mean of some 60 to 100 values of $u_x(r)$ the weight of the determinations for large values of $x$ is not very great and the curves become untrustworthy towards the right-hand side of the diagram. The reason is that there is a good deal of overlap of the $u_x(r)$; any change of phase occurring during a few periods will affect all the differences $a_{r+x} - a_r$ which enclose those periods. Thus when $x$ is large, the weight of $\bar{u}_x$ is much less than if it were the mean of 60 to 100 independent values of $u_x(r)$. On the left of the diagram the probable error of any ordinate is about 5 per cent.

The $\alpha$ term turns out to be small and, since it includes observational error in fixing the date of maximum, there is little, if anything, left to attribute to hypothesis (3). The maximum of *o* Ceti is flatter than that of $\chi$ Cygni, so that there is greater uncertainty in the observed date;

* Owing to the small weight of the determinations for large values of $x$ (see below), the values of $\alpha$ and $\epsilon$ are chosen so as to give the best representation up to about $x=20$.

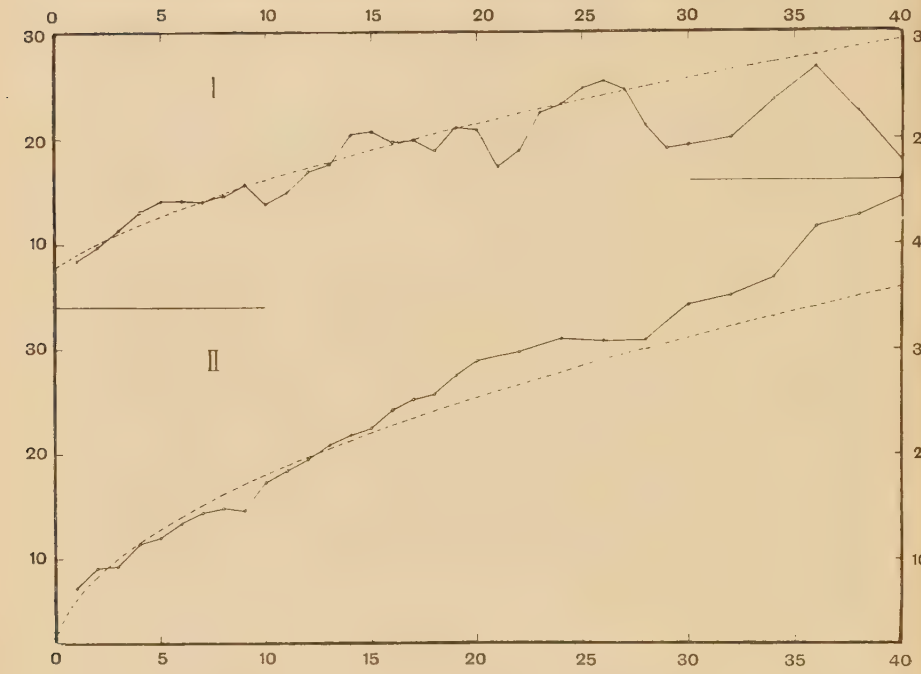


FIG. 1.—Curves of $\bar{u}_x$ for $o$ Ceti (I.) and $\chi$ Cygni (II.).

FIG. 2.—Curve for $o$ Ceti, showing a spurious periodicity.

and this seems to account adequately for the larger value of $\alpha$ found for the former.

The $\epsilon$ terms for the two stars (representing irregularities obeying hypothesis (1)) are very nearly in the same proportion as their periods. Thus the average deviation of an individual period of *o* Ceti from the mean is 1·35 per cent., and that of $\chi$ Cygni is 1·39 per cent. It would be interesting to discover whether this equality extends to other long-period variables. On the other hand it is possible that the larger value of $\epsilon$ is associated with the larger light-range rather than the longer period of $\chi$ Cygni.

5. Before discussing the significance (if any) of the deviations of the observed curves from the theoretical curves, we may indicate briefly the effects of other types of irregularity. Irregularities giving the term $\alpha$ may be called *temporary* since the phase difference at one maximum has no effect on, or correlation with, the phase difference at subsequent maxima. Those giving the term $\epsilon$ may be called *permanent* since the phase difference at one maximum is carried forward to all subsequent maxima, although it may be obliterated or enhanced by accidental irregularities incident subsequently. A third class may be called *repeated*, in which the phase difference grows proportionately to the number of periods elapsed except in so far as fresh irregularities supervene; to this class belong discontinuous changes of period, or more generally irregularities of type 2 (*a*) where there is a positive correlation between successive periods. The effect on the $\bar{u}_x$ curve can be found analytically. Since

$$u_x(r) = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_x,$$

where the $\epsilon$'s are the fluctuations of successive periods from the mean, we have

$$\left.\begin{aligned} \{u_x(r)\}^2 = \quad & \epsilon_1^2 + \epsilon_2^2 + \ldots \quad (x \text{ terms}) \\ & + 2\epsilon_1\epsilon_2 + 2\epsilon_2\epsilon_3 + \ldots \quad (x-1 \text{ terms}) \\ & + 2\epsilon_1\epsilon_3 + 2\epsilon_2\epsilon_4 + \ldots \quad (x-2 \text{ terms}) \\ & + \ldots \end{aligned}\right\} \quad \ldots (6)$$

If $r_1$ is the correlation coefficient between the fluctuations of successive periods, we have (by definition of $r_1$)

$$\text{mean value of } \epsilon_s\epsilon_{s+1} = r_1\mu^2,$$

where $\mu$ is the mean square value of $\epsilon_s$. If $r_2$ is the correlation coefficient between a period and that next but one preceding, and so on, we have from (6),

$$\text{mean value of } \{u_x(r)\}^2 = \mu^2(x + 2r_1(x-1) + 2r_2(x-2) + \ldots).$$

The average values $\bar{u}_x$, $\epsilon$, used in our work, will be related in practically the same way as the mean square values; hence

$$\bar{u}_x = \epsilon\sqrt{x} \cdot \sqrt{\left\{1 + \frac{x-1}{x}2r_1 + \frac{x-2}{x}2r_2 + \ldots\right\}} \quad \ldots (7)$$

If the irregularity consists of abrupt changes of period occurring not too frequently, the coefficients $r_1$, $r_2$, $r_3$, . . . will decrease very slowly. If it has more the form of a simple correlation (the correlation between $\epsilon_s$ and $\epsilon_{s+2}$ being no more than must automatically result from both being correlated to $\epsilon_{s+1}$) the coefficients will decrease rapidly. In either case the curve for sufficiently large values of $x$ approximates to the standard form $\bar{u}_x \propto \sqrt{x}$, the constant coefficient being $\epsilon\sqrt{(1 + 2r_1 + 2r_2 + \ldots)}$; as $x$ diminishes, the value of $\bar{u}_x$ will drop below the $\sqrt{x}$ curve.

A negative correlation will be indicated similarly by the opposite kind of deviation from a $\sqrt{x}$ curve, viz. if a $\sqrt{x}$ curve is fitted for large values of $x$, the observed $\bar{u}_x$ will rise above the curve as $x$ diminishes. A negative correlation is equivalent to a mixture of "temporary" and "permanent" irregularity, so that it is fairly well covered by our formula (4) which includes both types.

There remains the possibility of a periodic fluctuation of the time of maximum. This will be indicated by a sinuosity in the curve representing $\bar{u}_x$. For example, if the period of the fluctuation is 10 times the period of the light-change, the fluctuation will cancel out in forming $a_{r+10} - a_r$, $a_{r+20} - a_r$, etc., so that $\bar{u}_{10}$, $\bar{u}_{20}$, . . ., will be lower than neighbouring ordinates.

6. Bearing in mind that there is rather large accidental error for points on the right of the diagrams, the curves for o Ceti and $\chi$ Cygni appear to be fairly accordant with the hypothesis in § 2, viz. purely accidental fluctuations of period coupled with an $\alpha$ term which probably represents the observational errors in fixing the time of maximum. Applying the criterion found in § 5, there is no evidence of repeated fluctuations (positive correlations).

For o Ceti, however, the dip of the curve at $x = 10$, 21, 31, with intermediate maxima, suggests that part of the irregularity may be periodic, with period about 10 times that of the star, or 9 years. Whilst this requires us to make some reservation in claiming that the fluctuations are of the purely accidental type, we do not wish to imply that there are any strong grounds for thinking that this periodicity is genuine. We have in fact rather strong evidence that the sinuosity of curve I. is an accident arising from the incompleteness of the data. Rather by inadvertence, the curve in fig. 2 was plotted, showing the mean value of $u_x(r)$, *with attention to sign*, plotted against $x$. It is easily seen that for a long and complete series of observations this curve should coincide with the axis of $x$, since in forming the mean $u_x$ with attention to sign there is a general cancelling out of intermediate fluctuations, leaving only a few fluctuations at each end to contribute to the result. The curve, therefore, merely illustrates the chance effect of incompleteness of data (many maxima being unobserved). Comparison with curve I. in fig. 1 shows that the sinuosities of both curves correspond closely. It is fairly evident, therefore, that those of curve I. are to be attributed to the same chapter of accidents, and can have no physical importance.

7. The problem may perhaps be made clearer by a physical picture.

We may compare the outburst of light at maximum to the breaking of a wave or to a boiling over of the star—phenomena suggestive of some degree of irregularity. The boiling over relieves a strain, and afterwards there is a period of waiting until the strain mounts up sufficiently to cause another outburst. The waiting period must be reckoned as starting from the actual time of outburst (whether early or late), so that the epoch of the next maximum is determined partly by the epoch of the previous maximum and partly by the accidental circumstances which render the period of waiting longer or shorter than the average. The simplest hypothesis is that nothing else enters into the problem, and our curves indicate that this accords reasonably well with the observations. It would not have been surprising if the relief afforded by a quick or premature outburst differed somewhat from that afforded by a delayed outburst so that one or other of the two correlations (*2a*) or (*2b*) might occur. We find no evidence of these in the curves.

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*A Revision of Newcomb's Occultation Memoir.* By H. Spencer Jones, M.A., Sc.D., H.M. Astronomer. (Plate 1.)

In the year 1912 was published Newcomb's long-awaited memoir, "Researches on the Motion of the Moon," Part II.,* containing a derivation of the mean motion of the Moon and of other astronomical elements, based upon observations of occultations extending from 1672 to 1908. This memoir, the preparation of which had extended over thirty years, was completed during the author's last illness, and he was unable to see the work through the press.

In this memoir Hansen's Tables were used as a basis for computation, corrections being applied to take account of the principal terms in the longitude of the Moon omitted by Hansen, thereby reducing Hansen's elements to what Newcomb termed "the provisionally accepted theory." Some of the elements used were changed in the course of the reductions, as improved values became available. Newcomb contemplated that at some future date a revision of the work would be required, doubtless having in mind the completion of Brown's Tables of the Moon. His intention was to arrange the memoir in such a manner that no revision of the details of the work would be required, and that the discovery of any errors of computation and their corrections would be as easy as possible. He further stated that "In the reconstruction of the work most of the numbers, especially those pertaining to the equations of condition, can be used without revision" (p. 7), and "It will not be necessary even to repeat the computation of the

* *Astronomical Papers of the American Ephemeris*, 9, pt. i., 1912.

normal equations, especially as these have been so arranged as to facilitate any new combination that may be desirable " (p. 16).

Following upon the reduction and discussion of the Cape observations of occultations in the years 1880 to 1922,* the revision of Newcomb's occultation memoir was taken in hand four years ago. In view of certain discordances between the results given by the occultations and by meridian observations, as revealed by Brown's discussions, the reduction of Newcomb's provisionally accepted theory to the basis of Brown's theory seemed desirable in order to ascertain to what extent the discordances found were attributable to the incomplete representation of the Moon's motion, which formed the basis of Newcomb's discussion. For this purpose longitude corrections to take account of the principal terms of long and short period omitted by Newcomb were applied, and the terms depending upon the figure of the Earth were revised. This method is not so satisfactory as a complete recomputation on the basis of Brown's Tables of the place of the Moon at each occultation, but the work is much lighter, and it is hoped that no terms of importance have been omitted.

Early in the work it was found that there were many errors both in magnitude and sign in the coefficients of the equations of condition, and that the confidence expressed by Newcomb in their accuracy was not justified. In response to my request, Professor Eichelberger kindly forwarded the original copy for press, and a comparison with this showed that in only a few cases were the errors due to careless proof reading. The discovery of these errors necessitated a careful examination of the coefficients of the equations of condition, based on the position angles $m$, $m_1$, $m'$ given in chapter vii. of the memoir. Whilst it cannot be claimed that all the errors have been corrected, the more so as some of the corrections are themselves conjectural, it is hoped that any errors which remain will not, in view of the large amount of material under discussion, appreciably affect the derived results. Newcomb believed that the coefficients of the normal equations would not need revision, but the method of weighting employed in their formation (p. 214) is not explicitly stated, and the actual weights used from Group VI. onwards are not given. The recomputation of the normal equations, with an independent system of weighting, was therefore necessary. Attention may also be drawn to the fact that in §1 of chapter i. Newcomb draws attention to the value of occultations for deriving the coefficient of the principal term in the parallactic inequality, and the coefficients of the parallactic term are given in his equations of condition for the Groups VI. to XIII., covering the period 1820 to 1908; this term is nevertheless dropped from the normal equations without any reference to it. In the present revision the parallactic term is retained.

The purpose of the present communication is to summarise the results obtained, not to elaborate details of the computations, which will be published in the *Cape Annals*. It is sufficient to state here that the method of successive approximations was employed to derive the

* *Cape Annals*, **8**, pt. 8, 1925.

various corrections. In Newcomb's notation these are denoted and defined as follows :—

$\lambda$, the correction to the Moon's mean longitude.
$\kappa = -2e\delta\Pi$ where $\Pi$ is the longitude of the Moon's perigee and $e$ is the eccentricity of the Moon's orbit.
$i\theta = \sin i\delta\Omega$ where $\Omega$ is the longitude of the Moon's node.
$i$, the correction to the inclination of the Moon's orbit.
$b_0$, the correction to the Moon's tabular latitude.
$\alpha_0$, the mean correction to all the right ascensions of the stars, *i.e.* to the adopted position of the equinox.
$\delta_0$, the mean correction to the declinations of all the stars.
$\epsilon$, the correction to the tabular obliquity of the ecliptic.
P, the correction to the coefficient of the principal term in the parallactic inequality.

Of these quantities $\lambda$ is subject to fluctuations, $\kappa$, $i\theta$, $\alpha_0$, $\delta_0$, and $\epsilon$ are assumed to vary with the time, $i$, $b_0$, and P are assumed to be constant.

The method used was to derive preliminary values for the longitude corrections and then to determine the remaining constants for the several groups into which Newcomb divided his material (with the exception that P is omitted from Groups I.–V.). The variable quantities $\kappa$, $i\theta$, $\alpha_0$, $\delta_0$, and $\epsilon$ were then expressed in the form $a + b\mathrm{T}$, using the group values. In the normal equation for any one correction the preliminary values for the remaining corrections were substituted and a second approximation was obtained. Proceeding thus, when the approximation did not differ greatly from the final values, revised longitude corrections were derived and plotted ; smoothed values were read off and adopted as the final longitude corrections. Using these values, a further series of approximations for the remaining corrections were derived until successive approximations were practically identical.

In occultation observations systematic errors are liable to be introduced by the bright limb observations. The times of disappearance are liable to be observed too early and those of reappearance too late, except in the case of bright stars. In order to obviate errors of this nature, which would enter systematically into the determination of the parallactic inequality, observations of disappearance at the bright limb have been retained only in the case of stars of magnitude 3·2 or brighter ; observations of reappearance at the bright limb have been retained only in the case of stars of magnitude 2·1 or brighter.

The following corrections were applied to the longitudes, based on Hansen's Tables, as corrected by Newcomb :—

$$+ 0''\cdot13 \sin \{g - 3(\mathrm{V} - \mathrm{E}) + 180^\circ\}$$
$$+ 0''\cdot19 \sin \{g - 2(\mathrm{E} - \mathrm{J}) + 180^\circ\}$$
$$+ 0''\cdot08 \sin \{g - 2(\mathrm{E} - \Pi) + 6(\mathrm{V} - \mathrm{E}) - 2\mathrm{E} + 162^\circ\cdot6\}$$
$$+ 0''\cdot07 \sin \{g + (2\mathrm{M} - \mathrm{E}) + 223^\circ\cdot3\}$$
$$+ 0''\cdot07 \sin \{g - (2\mathrm{M} - \mathrm{E}) + 306^\circ\cdot3\}$$
$$+ 0''\cdot04 \sin \{g - 2\Pi - 3(2\mathrm{M} - \mathrm{E}) + 162^\circ\}$$
$$+ 0''\cdot02 \sin \{\mathrm{D} + \Pi - 2\mathrm{J} + 116^\circ\}$$

$$
\begin{aligned}
&+ 0''\cdot05 \sin \{D + \Pi - J + 180^\circ\cdot9\} \\
&+ 0\cdot28 \sin \{g + 2\omega + 180^\circ\} \\
&+ 0\cdot23 \sin \{2(D - g) - 2\omega\} \\
&+ 0\cdot16 \sin \{4(D - g) - 2(\omega - \omega')\} \\
&+ 0\cdot11 \sin \{4(D - g) - 2\omega - 2(\omega - \omega')\} \\
&+ 0\cdot12 \sin \{2(V - E) + 180^\circ\} \\
&+ 0\cdot10 \sin \{E - J + 174^\circ\} \\
&+ 0\cdot24 \sin \{-(8V - 13E) + 313^\circ\cdot9\} \\
&+ 1\cdot07 \sin \{-(18V - 16E - g) + 334^\circ\cdot4\} \\
&+ 0\cdot35 \sin \{-(18V - 16E - g) + (8V - 13E) + 33^\circ\cdot0\} \\
&+ 0\cdot84 \sin \{20^\circ\cdot2T + 31^\circ\cdot0\} \\
&+ 2\cdot25 + 1''\cdot36T - 2''\cdot42T^2 + 10''\cdot71 \sin (100^\circ\cdot7 + 140^\circ T) \\
&\qquad\qquad - 15''\cdot49 \sin (93^\circ\cdot9 + 132^\circ T).
\end{aligned}
$$

In the above

$g$ = the mean anomaly of the Moon.
$g'$ = ,, ,, ,, ,, Sun.
D = difference of mean longitudes of Moon and Sun.
$\omega$ = distance from Moon's ascending node to Moon's perigee.
$\omega'$ = ,, ,, ,, ,, ,, Sun's ,,
$\Pi$ = mean longitude of Moon's perigee.
V, E, M, J = mean longitudes of Venus, the Earth, Mars, and Jupiter respectively.
T = centuries elapsed since 1800·0.

The last five corrections are corrections of long period. The last correction is the correction to the Hansen-Newcomb mean longitude to reduce it to the longitude of Brown's Tables.

In addition, corrections to the figure of the Earth terms adopted by Newcomb were applied, to reduce them to the values computed by Brown's theory corresponding to an ellipticity of the Earth's figure of 1/297. These corrections were :

To the longitude + 1″·08 sin (Ω + 176°·8),
,, ,, perigee ($\kappa$) + 0″·04 sin (Ω + 115°),
,, ,, node ($i\theta$) + 0″·06 sin (Ω + 325°),

where Ω = the mean longitude of the Moon's ascending node.

Newcomb's reductions were based upon a value of the mean semi-diameter of 932″·58. This value is undoubtedly too small. For the revised discussion the value of 932″·70 derived from the discussion of the Cape occultations from 1880 to 1922 has been adopted.* This value is in close agreement with the determination of L. Struve,† 932″·65, and H. Battermann,‡ 932″·72. The corresponding corrections have been applied to the equations of condition.

The final values for the corrections, other than the longitude correc-

* *Cape Annals*, **8**, pt. 8, p. 40H, 1925.
† *Ast. Nach.*, **135**, 175, 1894.
‡ *Beob.-Ergeb. der König.-Sternwarte zu Berlin*, No. 13, p. 43, 1910.

tions, are given in Table I., the weight of each value being given by the subscript figures. The probable error corresponding to unit weight is that of a single occultation observation.

TABLE I.

*Final Values of Corrections.*

| oup. | Mean Date. | $\kappa$. | $i\theta$. | $i$. | $b_0$. | $a_0$. | $\delta_0$. | $\epsilon$. | P. |
|---|---|---|---|---|---|---|---|---|---|
| -V | 1720 | $+1''.51_{32}$ | $-0''.92_{11}$ | $''$. | $0''.00_{10}$ | $-0''.44_{1\cdot8}$ | $+0''.92_{3}$ | $+1''.19_{19}$ | $''$. |
| I | 1761 | $+1''.37_{8}$ | $-1''.07_{5}$ | $-0''.86_{2}$ | $-0''.75_{4}$ | $-4''.06_{0\cdot3}$ | $-1''.63_{1\cdot2}$ | $+0''.38_{5}$ | $-1''.54_{11}$ |
| II | 1792 | $+1''.35_{15}$ | $-0''.84_{10}$ | $+0''.09_{8}$ | $-0''.12_{14}$ | $-1''.32_{1\cdot4}$ | $+1''.51_{3}$ | $+0''.76_{7}$ | $-0''.88_{14}$ |
| III | 1811 | $+0''.54_{28}$ | $-0''.70_{8}$ | $-0''.95_{7}$ | $+0''.08_{13}$ | $-0''.95_{1\cdot3}$ | $-0''.10_{3}$ | $+1''.11_{7}$ | $-1''.06_{37}$ |
| X | 1829 | $+0''.77_{62}$ | $-0''.27_{23}$ | $-0''.28_{46}$ | $-0''.46_{53}$ | $-0''.22_{5}$ | $-0''.15_{8}$ | $+0''.26_{34}$ | $-0''.37_{95}$ |
| | 1848 | $+0''.29_{116}$ | $+0''.38_{44}$ | $-0''.40_{58}$ | $-0''.48_{96}$ | $-1''.47_{8}$ | $-0''.02_{18}$ | $-0''.18_{57}$ | $-0''.32_{155}$ |
| I | 1865 | $-0''.30_{110}$ | $-0''.08_{55}$ | $-0''.44_{63}$ | $-0''.57_{106}$ | $-0''.48_{9}$ | $+0''.11_{20}$ | $+0''.41_{61}$ | $-0''.24_{157}$ |
| II | 1882 | $-0''.51_{217}$ | $+0''.31_{91}$ | $-0''.49_{134}$ | $-0''.52_{210}$ | $-0''.83_{16}$ | $+0''.17_{33}$ | $+0''.31_{126}$ | $+0''.06_{355}$ |
| III | 1899 | $-0''.79_{372}$ | $+0''.88_{165}$ | $-0''.03_{231}$ | $-0''.51_{329}$ | $-0''.27_{34}$ | $+0''.16_{67}$ | $+0''.18_{190}$ | $-0''.12_{620}$ |

From these values were derived the following results for the corrections to the elements for 1850 and their centennial motions :—

| | P.E. unit weight. |
|---|---|
| $\kappa = +0''.03 \pm 0''.05 - (1''.56 \pm 0''.12)$ T | $\pm 1''.43$ |
| $i\theta = +0''.11 \pm 0''.07 + (1''.22 \pm 0''.16)$ T | $\pm 1''.25$ |
| $i = -0''.26 \pm 0''.06$ | $\pm 1''.33$ |
| $b_0 = -0''.49 \pm 0''.03$ | $\pm 0''.74$ |
| $a_0 = -0''.71 \pm 0''.13 + (0''.54 \pm 0''.29)$ T | $\pm 0''.93$ |
| $\delta_0 = +0''.16 \pm 0''.09 - (0''.10 \pm 0''.20)$ T | $\pm 0''.89$ |
| $\epsilon = +0''.30 \pm 0''.07 - (0''.37 \pm 0''.15)$ T | $\pm 1''.38$ |
| $P = -0''.173 \pm 0''.055$ | $\pm 2''.07$ |

In the above expressions T is expressed in centuries measured from 1850. The individual values in Table I. were combined with the weights obtained from the normal equations, and the probable errors given above are based upon these weights and upon a probable error for unit weight for each quantity which is derived from the residuals from the values computed by the above expressions.

The group values for each correction furnish a determination of the probable error attaching to unit weight. These values are given above. This quantity can also be determined from the residuals given by substitution of the final values of the constants in the equations of condition. If the above values of the probable error for unit weight are considered to be merely accidental variations from the same quantity, the adopted value for the probable error for unit weight will be

$$\pm 1''.25 \pm 0''.28.$$

The theoretical value of the probable error of the probable error based on eight determinations is

$$\pm \cdot 477r/\sqrt{7} = \pm 0''\cdot 23$$

for a probable error of $\pm 1''\cdot 25$. This value is in sufficiently close agreement with the value of $\pm 0''\cdot 28$ above to justify the conclusion that the greater part of the variation in the individual values of the p.e. is accidental in nature. A small portion of it may be due to errors of a quasi-systematic nature affecting the determinations of the several corrections.

The probable error for unit weight was derived from the equations of condition by substitution in 700 equations spread approximately uniformly throughout the series. The value obtained in this way from the squares of the residuals was $\pm 0''\cdot 94$, appreciably smaller than the previous value, although not differing from it by much more than the probable error of the latter. In order not to underestimate the probable errors attaching to the derived corrections, the larger value ($\pm 1''\cdot 25$) of the p.e. per unit weight has been adopted in what follows. It should, however, be mentioned that the distribution of residuals—as was pointed out by Newcomb and confirmed by the present revision—does not follow a normal error law. Small and large residuals are both relatively too frequent, intermediate residuals being relatively too few.

*The Longitude of the Perigee.*

The correction derived is a correction to Hansen's value for the longitude of the perigee, and has been expressed in the form $a + b\mathrm{T}$. But the longitude of the perigee contains also terms depending upon $\mathrm{T}^2$ and $\mathrm{T}^3$, the coefficients of which can be derived more accurately from theory than from observation. Brown's investigations indicate that, so far as these terms are concerned, Hansen's longitude of the perigee requires the correction

$$-0''\cdot 91\mathrm{T}^2 - 0''\cdot 008\mathrm{T}^3.$$

This correction should be made to Hansen's value before deriving the portion of the correction of the form $a + b\mathrm{T}$. We then derive

$$\kappa = 0''\cdot 00 \pm 0''\cdot 04 - (1''\cdot 512 \pm 0''\cdot 10)\mathrm{T},$$

corresponding to

$$d\Pi = 0''\cdot 00 \pm 0''\cdot 36 + (13''\cdot 77 \pm 0''\cdot 91)\mathrm{T}.$$

The longitude of the perigee for epoch 1850, according to Hansen and Brown, is as follows :—

Hansen, $\Pi = 99^\circ\ 51'\ 54''\cdot 74 + 11^r 109^\circ\ 2'\ 26''\cdot 30\mathrm{T} - 36''\cdot 19\mathrm{T}^2 - ''\cdot 037\mathrm{T}^3.$
Brown, $\Pi = 99^\circ\ 51'\ 56''\cdot 38 + 11^r 109^\circ\ 2'\ 39''\cdot 66\mathrm{T} - 37''\cdot 10\mathrm{T}^2 - ''\cdot 045\mathrm{T}^3.$

In so far as the first two terms are concerned, Brown's correction to Hansen is

$$+ 1''\cdot 64 + 13''\cdot 36\mathrm{T}.$$

The correction to Hansen derived here is

$$+ 0''\cdot00 + 13''\cdot77T,$$

and the correction to Brown's value for the longitude of perigee required by the occultations is

$$- 1''\cdot64 \pm 0''\cdot36 + (0''\cdot41 \pm 0''\cdot91)T.$$

The probable errors are a measure of the uncertainty of the occultation results only. Brown did not assign any values to the probable errors of his determination. It will be seen that the correction to the value at epoch will not be greatly in excess of the combined probable errors of the two determinations, and that the correction to the mean motion is in agreement with Brown's value well within the assigned limits of probable error. This agreement is satisfactory because the comparison between Newcomb's longitude for the perigee, as slightly modified by Brown, and Brown's value was

$$- 1''\cdot74 + 4''\cdot52T.$$

Brown * adds that "This is larger than the modern Greenwich meridian observations will admit, and a closer examination is therefore necessary in order to see whether an error of this magnitude is probable in the occultation results." The revision of the occultation results has succeeded in harmonising the mean motion derived from the two types of observation.

In the discussion of the Cape occultations for 1880 to 1922 a correction to Brown's longitude of perigee for epoch 1900 of amount $- 1''\cdot64 \pm 0''\cdot33$ was derived.† This correction is closely confirmed by the present discussion.

Adopting the theoretical values of the coefficients of the terms in $T^2$ and $T^3$, as derived by Brown, the longitude of the perigee given by the occultations is, for epoch 1850,

$$\Pi = 99^\circ\ 51'\ 54''\cdot74 + 11^r\ 109^\circ\ 2'\ 40''\cdot07T - 37''\cdot10T^2 - 0''\cdot045T^3.$$

*The Longitude of the Node.*

As in the case of the perigee, Hansen's longitude of the node should be corrected for errors in the terms depending upon $T^2$ and $T^3$, which can be derived from theory, before the corrections in Table I. are expressed in the form $a + bT$. The longitudes of the node for epoch 1850, according to Hansen and Brown, are as follows :—

Hansen, $\Omega = 146^\circ\ 13'\ 38''\cdot72 - 5^r\ 134^\circ\ 8'\ 51''\cdot41T + 8''\cdot20T^2 + ''\cdot007T^3$.
Brown, $\Omega = 146^\circ\ 13'\ 41''\cdot95 - 5^r\ 134^\circ\ 8'\ 38''\cdot70T + 7''\cdot47T^2 + ''\cdot008T^3$.

After correcting for the terms $- 0''\cdot73T^2 + ''\cdot001T^3$, we derive

$$i\theta = + 0''\cdot13 \pm 0''\cdot07 + (1''\cdot200 \pm 0''\cdot16)T,$$

* *M.N.R.A.S.*, **74**, 420, 1914. † *Cape Annals*, **8**, pt. 8, p. 37H, 1925.

corresponding to

$$d\Omega = + 1''\cdot45 \pm 0''\cdot80 + (13''\cdot40 \pm 1''\cdot78)\mathrm{T}.$$

The correction to Brown's longitude of the node is therefore

$$- 1''\cdot78 \pm 0''\cdot80 + (0''\cdot69 \pm 1''\cdot78)\mathrm{T},$$

the probable errors being a measure of the uncertainty of the occultation determination only. Brown does not give the probable error of his determination based on Greenwich meridian observations. It is satisfactory to note that Brown's value of the mean motion of the node is very closely confirmed. In addition, the two determinations of the longitude at epoch do not differ by much more than their combined probable error.

In the discussion of the Cape occultations, 1880 to 1922, a correction to Brown's longitude of the node for epoch 1900 of amount $+ 1''\cdot87 \pm 0''\cdot59$ was derived.* The present investigation gives a correction in the opposite sense. It may therefore be concluded that Brown's longitude at epoch is substantially correct.

Adopting the theoretical values of the coefficients of the terms in $\mathrm{T}^2$ and $\mathrm{T}^3$, as derived by Brown, the longitude of the node as given by the occultations is, for epoch 1850,

$$\Omega = 146^\circ\ 13'\ 40''\cdot17 - 5^r\ 134^\circ\ 8'\ 38''\cdot01\mathrm{T} + 7''\cdot47\mathrm{T}^2 + ''\cdot008\mathrm{T}^3.$$

### *The Inclination.*

The coefficient of the principal term in latitude in Hansen's theory, according to Cowell,† is $18461''\cdot65$. This value is accepted by Brown. Newcomb ‡ uses the value $18461''\cdot64$.

Cowell derives a correction of $- 0''\cdot30$ to Hansen's value. Brown § corrects Cowell's value by $+ 0''\cdot04$, and a further correction of $+ 0''\cdot07$, arising from the use for a certain period of an incorrect value of the obliquity of the ecliptic, was later applied. ‖ Brown's final value for the coefficient of the principal term in latitude is therefore

$$18461''\cdot40 \pm 0''\cdot03.$$

The correction to Hansen's value, obtained above, is $- 0''\cdot26 + 0''\cdot05$, giving for the coefficient the value

$$18461''\cdot39 \pm 0''\cdot05.$$

The agreement is all that could be desired.

Newcomb assigned to his determination from the occultations of the coefficient of the principal term in latitude a probable error of $\pm 0''\cdot19$, and for this reason Brown considered that it was not entitled to great

* *Cape Annals*, **8**, pt. 8, p. 38H, 1925.
† *M.N.R.A.S.*, **65**, 564, 1905.
‡ "Researches on the Motion of the Moon," pt. ii., p. 224, 1912.
§ *M.N.R.A.S.*, **74**, 561, 1914.
‖ *M.N.R.A.S.*, **75**, 507, 1915.

weight. It is not apparent why so large a probable error was assigned. From the discordances between the values derived from the separate groups, together with the weights adopted by Newcomb, a probable error of $\pm$ 1″·27 for unit weight is obtained, corresponding to a probable error for the weighted mean value of $\pm$ 0″·05, in agreement with the error given above. The probable error of $\pm$ 0″·19 given by Newcomb would correspond to a probable error for unit weight of nearly $\pm$ 5″, an impossibly high value for occultation observations.

### *The Latitude Term.*

Hansen applied a constant correction of $-$ 1″·00 to the latitude of the Moon on account of a supposed non-coincidence of the centres of gravity and figure of the Moon. Newcomb removed this correction, remarking that "It seems probable that the seeming necessity for this correction arose partly because Hansen used too small a value for the Moon's parallax, and partly from systematic errors in the observations of declination with which his comparisons were made."* Newcomb included, however, a term $b_0$ in the equations of condition based on the occultation observations to allow for such a term, if it actually existed.

The values for $b_0$ given in Table I. show most convincingly the reality of the correction. The first four values, which have small weight, are somewhat discordant, but the last five groups which have high weight are remarkably consistent. The weighted mean value for all the groups is

$$-\ 0''{\cdot}49 \pm 0''{\cdot}03.$$

If this term is due to inequalities in the outline of the Moon's disc, which are more or less systematic in their character, a difference in the values of $b_0$ for immersion and emersion phenomena might be expected. The immersion and emersion observations for the last five groups were therefore separated, and the following values of $b_0$ were obtained.

TABLE II.

*Values of Latitude Correction.*

| Group. | Emersions. | | Immersions. | |
|---|---|---|---|---|
| | $b_0$ | Weight. | $b_0$ | Weight. |
| | ″ | | ″ | |
| IX | −0·41 | 13 | −0·47 | 60 |
| X | +0·14 | 25 | −0·70 | 71 |
| XI | −0·21 | 28 | −0·70 | 78 |
| XII | −0·35 | 70 | −0·61 | 140 |
| XIII | −0·34 | 105 | −0·58 | 224 |
| Mean | −0·28 | 241 | −0·61 | 573 |

The correction is therefore systematically larger for the east limb (immersions) than for the west limb (emersions).

* "Researches on the Motion of the Moon," pt. ii., p. 38, 1912.

From a discussion of the meridian observations, Brown found that, after removing Hansen's empirical term of 1″·00 in the latitudes, a correction to Hansen's tabular latitude of − 0″·51 was required.* The mean value derived above is in almost exact agreement with Brown's value. Brown investigated whether the correction derived from the meridian observations could be explained by instrumental or other errors of observation, and concluded that this was not possible.

Brown concluded that the most probable explanation is to be found in local irregularities of the Moon's limb. The constant term, − 0″·51, found from the meridian observations, implies that the centre of the observed polar axis is below the centre of mass by about half a second ; the term, − 0″·61, derived from the immersions, implies that there is a large average depression of the N.E. to E. portion of the limb as compared with the S.E. to E. portion ; the term, − 0″·28, derived from the emersions, implies that there is a smaller average depression of the N.W. to W. portion as compared with the S.W. to W. portion. Brown points out that these conclusions are in accordance with the papers of Franz and Hayn ; Franz's map shows that the Moon's equator approximately divides the low from the high ground, the high ground being in the southern part.

It may be mentioned, in confirmation, that Przybyllok † and Battermann ‡ have shown that the probable error of an occultation observation is appreciably reduced when limb corrections based on Hayn's observations are applied in the reductions. This procedure should evidently be followed if the highest accuracy is desired. Tables for the application of the corrections are given by Przybyllok.†

### *The Obliquity of the Ecliptic.*

The observations give for the correction to the adopted obliquity of the ecliptic, which is Newcomb's value,

$$+ 0''{\cdot}30 \pm 0''{\cdot}06 - (0''{\cdot}37 \pm 0''{\cdot}14)\mathrm{T}.$$

Chief interest attaches to the correction to the secular change of the obliquity. The correction to the value at epoch is bound up with the value of the adopted ellipticity of the figure of the Earth and with the declination system. The value derived above cannot be interpreted, as it stands, as a correction to the obliquity alone. The term dependent upon T can only be interpreted, on the other hand, as due to an error in the adopted value of the secular change of the obliquity.

Newcomb's value for the secular centennial change of the obliquity

* *M.N.R.A.S.*, **74**, 160, 566, 1914.

† "Das Profil der Randpartion des Mondes," *Mitt. d. Grossh. Sternwarte zu Heidelberg*, No. xi., 1908.

‡ "Beitrag zur Bestimmung der Mondbahn und des Mondhalbmessers," *Beob.-Ergeb. der König. Sternwarte zu Berlin*, No. 13, 1910.

for 1850 is $-46''\cdot84$. The value given by the occultations is therefore $-47''\cdot21$.

Brown * derived from the Greenwich meridian observations of the Moon the value $-47''\cdot28$ or $-47''\cdot21$, according as the doubtful 28 P term is excluded or included. This value was corrected subsequently to $-47''\cdot09$, the term of period 28 P being omitted.† This is Brown's final value from the meridian observations of the Moon.

Newcomb derived the secular change from meridian observations of the Sun, Mercury, Venus, and Mars, supplemented by observations of the transits of Mercury and Venus. The corrections to Leverrier's value which he obtained are as follows :—‡

| | Correction. | Adopted Weight. |
|---|---|---|
| Sun | $+0''\cdot48$ | 5 |
| Mercury A | $+0''\cdot27$ | |
| ,, B | $+0''\cdot39$ | 1 |
| Venus A | $+0''\cdot29$ | |
| ,, B | $+0''\cdot32$ | 1 |
| Mars | $+1''\cdot03$ | $\frac{1}{2}$ |
| Adopted mean | $+0''\cdot48$ | |

The A values for Mercury and Venus depend upon meridian observations alone. The B values include also the observations of the transits of these planets.

Newcomb, however, did not adopt the value so obtained, but employed instead the theoretical value computed from his adopted masses of the planets.§ The actual correction to Leverrier's secular change of the obliquity adopted by Newcomb and used in his tables is $+0''\cdot76$, the values of Leverrier and Newcomb for 1850 being respectively $-47''\cdot594$ and $-46''\cdot838$. A correction of $-0''\cdot28$ to Newcomb's adopted value would therefore be required to bring it into accordance with the observations of the Sun and planets used by him.

The material discussed by Newcomb extended only to 1892. The principal weight is given by the solar observations, the longest series of which is the Greenwich series. The early observations of Bradley and Maskelyne are discordant and of small weight. From the Greenwich observations of the Sun obtained during the period 1812 to 1927 I have derived a correction of $-0''\cdot30 \pm 0''\cdot16$ to Newcomb's value of the secular change of the obliquity.

* *M.N.R.A.S.*, **74**, 564, 1914.
† *M.N.R.A.S.*, **75**, 507, 1915.
‡ *Astronomical Constants*, p. 88.
§ "It is also to be noted that the secular variations of . . . and $\varepsilon$ in the definitive theory and tables are those computed from the adopted masses of the planets" (*Astronomical Constants*, p. 89).

The results may therefore be summarised as follows :—

| | Correction to Newcomb's Value. |
|---|---|
| Observations discussed by Newcomb | $-0''.28 \pm 0''.16$ |
| Greenwich Sun observations, 1812–1927 | $-0''.30 \pm 0''.16$ |
| Greenwich Moon observations | $-0''.25 \pm 0''.14$ |
| Occultation observations | $-0''.37 \pm 0''.14$ |

The four results are concordant within the range of the probable error of any single determination, and indicate a correction to Newcomb's value of the secular change of $-0''.30 \pm 0''.08$.

The theoretical value of the secular change of the obliquity is given by Newcomb * as

$$-46''.65 - 0''.21\nu - 28''.3\nu' - 0''.7\nu'''$$

where $\nu$, $\nu'$, $\nu'''$ are corrections to the following values of the masses of Mercury, Venus, and Mars respectively :—

| | |
|---|---|
| Mercury | 1/7,500,000 |
| Venus | 1/410,000 |
| Mars | 1/3,093,500. |

The theoretical value of the secular change of the obliquity depends therefore largely upon the assigned mass of Venus, and Newcomb's value can be reconciled with observation by modifying the mass of Venus which he adopted.

With de Sitter's value for the mass of Mercury of 1/8,000,000 (corresponding to $\nu = -0.06 \pm 0.20$) we obtain

$$28.3\nu' = +0.50 \pm 0.09,$$

whence

$$\nu' = +0.0176 \pm 0.0032.$$

The resulting reciprocal of the mass of Venus is therefore

$$402{,}910 \pm 1270.$$

From observations of Mars, Ross derived the value †

$$403{,}490 \pm 2400.$$

From the Greenwich observations of the Sun I obtained the value ‡

$$404{,}700 \pm 800.$$

It may therefore be concluded that the mass of Venus, which suits the observations of the Sun and of Mars, will lead to a theoretical value of the secular change of the obliquity of the ecliptic which agrees with the observed value within the probable error of its determination.

* *Astronomical Constants*, p. 108.
† *Astronomical Papers of the American Ephemeris*, **9**, pt. ii., p. 261, 1917.
‡ *M.N.R.A.S.*, **86**, 434, 1926.

The correction to the obliquity at epoch 1850 derived from the occultations, $+ 0''\cdot30 \pm 0''\cdot06$, does not agree with the value, $+ 0''\cdot08 \pm 0''\cdot06$, which I derive from the Greenwich solar observations from 1821 to 1927, though both indicate a correction to Newcomb in the same sense. Systematic errors may enter into either series. The transit-circle determination may be affected by errors of a seasonal nature or by the use of an incorrect value of the refraction constant. The occultation determination is bound up with the ellipticity of the Earth's figure and with the adopted declination system, and cannot be regarded as giving a determination of the true value of the obliquity. The final correction derived by Brown * from the Greenwich meridian observations of the Moon is almost midway between the above two values, being $+ 0''\cdot17$ at epoch 1850, but is based upon the value 1/294 for the ellipticity of the Earth's figure.

*The Ellipticity of the Earth's Figure and the Figure of the Moon.*

The theoretical values for epoch 1850 of the sidereal mean motions per century of the Moon's perigee and node, computed with the value 1/297 for the ellipticity of the Earth's figure, together with the value derived from the Greenwich meridian observations of the Moon by Brown and from the present discussion of the occultations, are as follows :—

| | Perigee. | Node. |
|---|---|---|
| Theoretical value | $+14643521''$ | $-6967931''$ |
| Meridian observations | $+14643535$ | $-6967944$ |
| Occultations | $+14643535$ | $-6967943$ |

The theoretical values are as computed by Brown, with the addition of the relativity corrections, $+ 1''\cdot97$ per century for the perigee and $+ 1''\cdot91$ per century for the node.† The values derived from observation are based on the value $50''\cdot2500$ for the precessional constant.

The theoretical values are the sums of a large number of terms, each of which was computed to the nearest integer only and may therefore be $0''\cdot5$ in error. The mean error of the sum is assumed by de Sitter to be about $\pm 2''$, corresponding to a probable error of about $\pm 1''\cdot3$. The probable errors of the determinations from the meridian observations were not precisely determined by Brown, but are of the same order of magnitude. From the occultation observations the probable errors are $\pm 0''\cdot9$ for the motion of the perigee and $\pm 1''\cdot8$ for the motion of the node. It will be apparent that a recomputation of the theoretical values of the motions to a higher degree of accuracy is greatly to be desired.

The figure of the Earth terms in the mean motions per century of the perigee and node, according to Brown's theory, are

$$d\Pi = [5\cdot5907]J, \qquad d\Omega = [5\cdot5620]J$$

* *M.N.R.A.S.*, **75**, 508, 1915. † De Sitter, *M.N.R.A.S.*, **77**, 172, 1916.

where $$J = 1/k - \cdot 0017287,$$

$k$ being the reciprocal of the ellipticity of the Earth's figure.

With $k = 297$ as basic value we derive

$$d\Pi = + 638''\cdot 3 \quad - 4''\cdot 42\delta k$$
$$d\Omega = - 597''\cdot 4 \quad + 4''\cdot 13\delta k$$

as the figure of the Earth terms for ellipticity $1/(297 + \delta k)$.

If, following Newcomb and Brown, the discordance between theory and observation is attributed wholly to error in the adopted ellipticity, we derive corrections to the ellipticity as follows :—

From the motion of the perigee :

$$- 4\cdot 42\delta k = + 14, \qquad \delta k = - 3\cdot 2.$$

From the motion of the node :

$$+ 4\cdot 13\delta k = - 13, \qquad \delta k = - 3\cdot 1,$$

corresponding to an ellipticity of

$$1/293\cdot 9.$$

The agreement is excellent, but is fortuitous. This method of deriving the ellipticity implies that all the terms which contribute to the theoretical values of the motions of the perigee and node rest on a firm theoretical foundation, with the exception of the uncertainty in the terms due to the figure of the Earth, arising from the uncertainty in the value of the ellipticity to be used. As pointed out by de Sitter,* the terms depending upon the figure of the Moon, adopted by Brown, are uncertain, in view of the fact that they depend only partially on actually determined constants.

Brown's Figure of the Moon terms depend upon the assumption that the Earth and Moon are of similar constitution, so that $C'/M : C/E$ in the ratio of the squares of their diameters.†

We proceed to examine to what extent Brown's assumption is consistent with the occultation results. We assume that the reciprocal of the ellipticity of the Earth is $297 + \delta k$, so that the figure of the Earth terms in the motions of the perigee and node are as given above. Removing the terms depending upon the figure of the Moon adopted by Brown, viz. $+ 3''$ for the perigee and $- 14''$ for the node, we must adopt, in order to reconcile the theoretical with the observed values, for the terms depending on the figure of the Moon :—

$$+ 17'' + 4''\cdot 42\delta k \text{ for the perigee}$$
$$- 26'' - 4''\cdot 02\delta k \quad ,, \quad \text{node.}$$

Brown ‡ gives the expression for the ratio of the motions depending

* De Sitter, *Proc. Acad. Amsterdam*, **17**, 1309, 1915.

† "Theory of the Motion of the Moon," pt. v., chap. xiii. (*Mem. R.A.S.*, **59**, pt. i., p. 81, 1908).

‡ *Loc. cit.*, p. 81.

upon the figure of the Moon and the figure of the Earth in terms of constants $\mu'$, $\mu''$ where

$$a^2\mu' = \frac{3}{2M}\left(C' - \frac{A' + B'}{2}\right); \qquad a^2\mu'' = \frac{3}{4M}(B' - A')$$

where M denotes the mass of the Moon, A', B', C' the moments of inertia. Proceeding in Brown's manner, but introducing the more convenient constants employed by de Sitter, viz.

$$J' = \frac{3}{2}\frac{2C' - A' - B'}{Mb'^2}; \qquad K' = \frac{3}{2}\frac{B' - A'}{Mb'^2}$$

where $b'$ denotes the largest radius of the Moon, we derive as the theoretical expressions of the figure of the Moon terms in the motions of the perigee and node :—

$$d\Pi = +(37900'' - 262''\delta k)\ J' - (99700'' - 691''\delta k)K'$$
$$d\Omega = -(45700'' - 316''\delta k)\ J' - (22900'' - 158''\delta k)K'.$$

We equate these to the observed values, viz.

$$d\Pi = +17'' + 4''\cdot42\delta k \pm 1''\cdot8$$
$$d\Omega = -26'' - 4''\cdot02\delta k \pm 2''\cdot4.$$

The assigned probable errors include the uncertainty both in the theoretical and in the observed values.

From these equations we derive

$$J' = +\cdot000550 + \cdot000096\delta k \pm \cdot000045$$
$$K' = +\cdot000038 - \cdot000009\delta k \pm \cdot000023.$$

The ratios

$$\alpha = \frac{C' - B'}{A'}; \qquad \beta = \frac{C' - A'}{B'}; \qquad \gamma = \frac{B' - A'}{C'}$$

appear in the theory of the libration of the Moon. We may take, with sufficient accuracy, $\beta = \alpha + \gamma$, so that only two of the constants require to be determined. These are generally introduced in the form $\beta$ and $f = \alpha/\beta$.

We have

$$J' + \frac{1}{2}K' = \frac{3}{2}\frac{C' - A'}{Mb'^2}$$
$$= \frac{3}{2}\frac{C'}{Mb'^2}\frac{C' - A'}{B'} = g'\beta.$$

Brown's assumption, to derive the figure of the Moon terms, was in effect that $g'$ has the same value as the corresponding quantity $g(=\cdot502)$ for the Earth. To determine $g'$ we must first determine $\beta$, which can be derived from the mean inclination of the equator to the ecliptic. De Sitter * derives in this way for $\beta$ the value

$$\beta = +0\cdot000626 \pm \cdot000002.$$

* *Proc. Acad. Amsterdam*, 17, 1312, 1915.

The resulting value of $g'$ is then

$$g' = +\cdot909 + \cdot147\delta k \pm \cdot08.$$

The value of $g'$ will equal that of $g = 3C/2Eb^2$ for the Earth if

$$+\cdot147\delta k = -\cdot407$$
$$\delta k = -2\cdot8$$

corresponding to an ellipticity of 1/294·2. This is, of course, closely equal to the value derived by accepting Brown's terms for the figure of the Moon and attributing the discordance between the observed and computed values entirely to the ellipticity of the Earth.

According to Clairaut's theory we must have

$$\frac{3}{5} > g' > 0,$$

the former value corresponding to the case of homogeneity, the latter to the concentration of the whole mass at the centre. The observed value does not fall within these limits unless $\delta k$ is negative; if it does not, the density cannot decrease from the centre outwards, but must increase at some point. It therefore follows, in view of the probable error of $g'$, that a value of 297 for the reciprocal of the ellipticity is improbable, but that a value of 296 is not impossible and a value of 295 would be reasonable.

We have further

$$f = \frac{\alpha}{\beta} = \frac{C' - B'}{C' - A'} = \frac{J' - \frac{1}{2}K'}{J' + \frac{1}{2}K'},$$

whence

$$f = +0\cdot933 + \cdot026\delta k \pm 0\cdot04.$$

It will be seen that $f$ does not depend greatly upon $\delta k$. $f$ is usually determined from certain small terms in the libration in longitude; the derived values vary considerably. Franz and Stratton obtained 0·49 and 0·50 respectively from the discussion of Schlüter's observations. Hayn obtained a value of 0·75 and, later, one of 0·85, and attributed the smaller values to errors in the adopted radius vector of Mösting A. Völkel obtained 0·78.

The determination of $f$ from the motions of the lunar perigee and node should be free from errors of a systematic nature. We have seen above that the meridian and occultation observations give motions which are in excellent agreement. The principal term in the libration in longitude, on the other hand, depends upon the sine of the mean anomaly of the Sun, and determinations of $f$ from the libration are therefore liable to be affected by systematic errors in the observations with a period of a year. It may be concluded that, whatever value is adopted for the ellipticity of the Earth's figure between the limits 1/293 and 1/297, the value of $f$ derived from the motion of the lunar perigee and node is nearer to unity than to $\frac{1}{2}$. If it is assumed that the Moon is in hydrostatic equilibrium, the value of $f$ can be determined

from theoretical considerations and is found to be exactly 0·25 * as compared with the observed value of 0·93. The hydrostatic equilibrium theory requires also, at the same time, that $\beta$ should have the value 0·0000375, whereas observations give the much larger value 0·000626. It has, in fact, long been known that the theory is incompatible with the observed data on the figure of the Moon.

The results of this discussion may be summarised as follows :—

If $1/297 + \delta k$ is the ellipticity of the Earth's figure, the terms in the mean motion of the perigee and node due to the figure of the Moon are

$$d\Pi = + 17'' + 4''\cdot42\delta k$$
$$d\Omega = - 26'' - 4''\cdot02\delta k.$$

The value of $g' = \frac{3}{2}C'/Mb'^2$ for the Moon is

$$g' = 0\cdot909 + \cdot147\delta k.$$

The value of $f = (C' - B')/(C' - A')$ is

$$f = +\cdot933 + \cdot026\delta k.$$

The values of $\alpha$, $\beta$, $\gamma$ are

$$\alpha = \frac{C' - B'}{A'} = + \cdot000584 - \cdot00001 6\delta k$$

$$\beta = \frac{C' - A'}{B'} = + \cdot000626$$

$$\gamma = \frac{B' - A'}{C'} = + \cdot000042 + \cdot00001 6\delta k.$$

These results alone will not enable $\delta k$ to be assigned. The ellipticity of the figure of the Earth must be otherwise determined. From a discussion of the various determinations, de Sitter has concluded that " None of the other determinations is equal in accuracy to, or can throw a doubt on, the determination from the constant of precession. We must therefore adopt as final value of the compression the result of this determination, viz. $1/\epsilon = 295\cdot96 \pm 0\cdot20$." †

If we adopt 1/296 for the ellipticity, $\delta k = - 1\cdot0$. The terms in the motions of the perigee and node due to the figure of the Moon are then

$$d\Pi = + 21'', \qquad d\Omega = - 30'',$$

as compared with the values + 3″ and − 14″ respectively adopted by Brown. We shall then have also

$$g' = 0\cdot76 \pm 0\cdot08,$$

---

* Jeffreys, *The Earth*, p. 201. The theoretical value for $(B' - A')/(C' - A')$ is $\frac{3}{4}$, whence $f = (C' - B')/(C' - A')$ is 0·25. Jeffreys incorrectly compares the observed value of $f$ with the former quantity, obtaining an apparent agreement between theory and observation which does not exist.

† " On Isostasy, the Moments of Inertia, and the Compression of the Earth," *Proc. Acad. Amsterdam*, **17**, 1295, 1915.

as compared with the value $g = \cdot 50$ for the Earth.

$$f = + 0\cdot 91 \pm 0\cdot 04$$
$$\alpha = + \cdot 000600\,; \qquad \beta = + \cdot 000626\,; \qquad \gamma = + \cdot 000026.$$

The observed and theoretical motions of the perigee and node are in agreement.

Newcomb used, in addition to the motions of the perigee and node, the coefficient of the sin $a$ term in latitude to derive the ellipticity of the Earth's figure. Owing to the fact that errors in the adopted obliquity, in the declination system, or in the adopted mean horizontal parallax all enter into this term, this procedure is not legitimate. Brown reversed the argument and, regarding the errors in the declination system and in the adopted parallax as small, derived a correction to the obliquity at the mean epoch.

The coefficient of the sin $a$ latitude term, in Brown's theory, corresponding to an ellipticity of 1/297 is $- 8''\cdot 31$, of which $- 8''\cdot 11$ is due to the figure of the Earth and $- 0''\cdot 20$ is due to the action of the planets, etc.

The change to an ellipticity of 1/294 would lead to a correction to the coefficient of the principal latitude term of amount

$$- 6 \times 8\cdot 11/297 = - 0''\cdot 16.$$

The coefficient used in Hansen's theory is $- 8''\cdot 27$, to which we have derived the correction $+ 0''\cdot 30 - 0''\cdot 37\mathrm{T}$, making the observed value

$$- 7''\cdot 97 - 0''\cdot 37\mathrm{T}.$$

But the theoretical value, corresponding to ellipticity 1/294, is

$$- 8''\cdot 31 - 0''\cdot 16 = - 8''\cdot 47,$$

whence the resulting correction to the obliquity would be

$$+ 0''\cdot 50 - 0''\cdot 37\mathrm{T}.$$

Not much weight can be given to the term independent of T, because it remains affected by errors in the declination system and in the adopted horizontal parallax. It does not, however, support the value 1/294 of the ellipticity. The correction found to the value of the obliquity accords better with an ellipticity of 1/297.

*The Right Ascension System and the Equinox Correction.*

The term in $a_0$ gives the mean correction to all the right ascensions of the stars and hence to the position of the equinox. Newcomb regarded the correction obtained in this way as entitled to considerable weight. He states that " I regard this correction to the equinox as probably more reliable than any heretofore obtained, so far at least as the centennial

motion is concerned. As has already been explained, it rests on the principle that the equinox can be determined from the Moon as well as from the Sun, while the probable error of the determination is much less. In the case of the equinoxes determined from the Sun, we have the unavoidable difficulties arising from the effect of the diurnal change of temperature on the instrument and the surroundings, and the uncertainty as to the personal equation of the observer. The latter is worthy of special consideration. Before the existence of the magnitude equation was known, it was necessarily assumed that there would be no personal equation between the observations of the Sun and of a star. Now, if we suppose, as we well might do, that the personal equation is the same for the Sun's limb and a star of the first magnitude, there would then be an equation of $0^s{\cdot}05$ between the Sun and a star of the seventh magnitude." *

The method, in principle, is sound, but the weight of the derived correction is relatively small. As will be seen from Table I., the corrections for the several groups run very irregularly and the weights for the early groups are small. The final result is therefore somewhat sensitive to the weights adopted. Newcomb, in his discussion, overweighted the earlier groups relatively to the later ones.

The final correction which we have obtained is

$$\begin{aligned} &-0''{\cdot}71 \pm 0''{\cdot}17 + (0''{\cdot}54 \pm 0''{\cdot}39)\mathrm{T} \\ = &-0^s{\cdot}047 \pm 0^s{\cdot}011 + (0^s{\cdot}036 \pm 0^s{\cdot}026)\mathrm{T}. \end{aligned}$$

This result can be strengthened if it is assumed that modern determinations of the equinox correction, based on meridian observations secured with an impersonal micrometer, can be assumed to be free from systematic error. The following determinations are available :—

| | | | s |
|---|---|---|---|
| Cape, 1907–11 | Observations of | Sun | − 0·064 |
| ,, ,, | ,, ,, | Mercury | − ·048 |
| ,, ,, | ,, ,, | Venus | − ·037 |
| ,, 1912–16 | ,, ,, | Sun | − ·053 |
| ,, ,, | ,, ,, | Mercury | − ·046 |
| ,, ,, | ,, ,, | Venus | − ·040 |
| Greenwich, 1906–14 | ,, ,, | Sun | − ·046 |
| ,, 1915–21 | ,, ,, | Sun | − ·070 |
| ,, 1922–27 | ,, ,, | Sun | − ·057 |
| ,, 1915–23 | ,, ,, | Venus | − ·058 |
| Washington, 1903–11 | ,, ,, | Sun | − ·052 |
| ,, 1911–18 | ,, ,, | Sun | − ·081 |

We adopt the following mean values :—

| | | |
|---|---|---|
| 1909·0 | − 0″·74 | weight 700 |
| 1916·0 | − 0″·94 | ,, 900 |
| 1925·0 | − 0″·85 | ,, 250 |

* "Researches on the Motion of the Moon," pt. ii., p. 222, 1912.

When these results are incorporated with the occultation results the equinox correction which results is :

$$\alpha_0 = -0''.65 \pm 0''.15 - (0''.30 \pm 0''.23)\frac{t - 1850}{100}$$

$$= -0^s.043 \pm 0^s.010 - (0^s.020 \pm 0^s.015)\mathrm{T}.$$

The correction at epoch 1850 is not appreciably altered, but the correction to the mean motion has changed sign. In both determinations this correction is of the same order as its probable error.

From the Greenwich observations of the Sun * I derived as the correction to the motion of the equinox as adopted by Newcomb,

$$+0^s.014 \pm 0^s.012,$$

a value which is again of the same order as its probable error.

The modern meridian determinations of the equinox correction, based on observations with the impersonal micrometer, appear to be free from systematic error, in view of the concordant corrections obtained from the Sun, Mercury, and Venus, and from observations with different instruments and at different observatories. Adopting, then, $-0''.71$ as the correction for 1850 given by the occultations, a small negative correction to the mean motion is necessary to give agreement with the modern meridian determinations.

Eichelberger's determination † of a correction to the mean motion of $-0^s.057$ per century is not supported. It is based on the assumption that Newcomb's equinox was correct for the mean epoch of its determination, about 1840. The occultation results indicate that this assumption is not valid.

*The Equatorial Correction to Newcomb's Declination System.*

The correction, $\delta_0$, gives the mean correction to the declination system of Newcomb's catalogue, which has been used for the star positions, in the neighbourhood of the equator.

The correction which has been derived is

$$\delta_0 = +0''.16 \pm 0''.12 - (0''.10 \pm 0''.29)\mathrm{T}.$$

Both terms of $\delta_0$ are therefore uncertain by practically their entire amount.

This result may be compared with that derived by Ross ‡ in the course of the derivation of new elements of Mars. Ross obtained the correction

$$+0''.09 \pm 0''.09 + (0''.02 \pm 0''.25)\mathrm{T}.$$

* *M.N.R.A.S.*, **86**, 438, 1926.
† *Astronomical Papers of the American Ephemeris*, **10**, pt. i., p. 21, 1925.
‡ *Ibid.*, **9**, pt. ii., p. 265, 1917.

From the discussion of the observations of Jupiter and Saturn Hill * obtained as correction to the system of Boss

$$+ 0''\cdot01 \pm 0''\cdot10 + (0''\cdot45 \pm 0''\cdot35)\mathrm{T}.$$

The reduction to Newcomb † at the equator is

$$+ 0''\cdot10 + 0''\cdot36\mathrm{T},$$

and Hill's value then becomes

$$- 0''\cdot09 \pm 0''\cdot10 + (0''\cdot09 \pm 0''\cdot35)\mathrm{T}.$$

The three determinations from Jupiter and Saturn, from Mars, and from the occultations are reasonably concordant and indicate that the correction required by Newcomb's declination system at the equator is very small. Newcomb's proper motions in declination in the equatorial region would appear also to be systematically more correct than those of Boss. The weighted mean of the three determinations gives the correction

$$\delta_0 = + 0''\cdot04 \pm 0''\cdot06 + (0''\cdot00 \pm 0''\cdot16)\mathrm{T}.$$

The corrections to Newcomb's proper motions in declination, found by Eichelberger,‡ are not confirmed. These amount to $+ 0''\cdot5$ per century at the equator.

*The Parallactic Inequality and the Solar Parallax.*

The correction to the coefficient of the principal term in the parallactic inequality which we have derived is

$$\mathrm{P} = - 0''\cdot173 \pm 0''\cdot033.$$

Newcomb applied a correction of $+ 1''\cdot60$ to Hansen's coefficient. The coefficient used in Hansen's tables is, according to Battermann, § $- 126''\cdot45$. Newcomb's basic value is therefore $- 124''\cdot85$.

The final value, derived from the occultations, is therefore

$$- 125''\cdot023.$$

The coefficient can be expressed in the form

$$- \mathrm{P} = \mathrm{C} \cdot \frac{a}{a'} \cdot \frac{\mathrm{E} - \mathrm{M}}{\mathrm{E} + \mathrm{M}'}$$

where E, M denote the masses of the Earth and Moon, and $a$, $a'$ are the semi-axes major of the orbits of the Earth and Moon, which, for undisturbed elliptic motion, correspond to the sidereal periods of revolution. C is a constant which can be computed theoretically.

* *Astronomical Papers of the American Ephemeris*, **7**, pt. i., p. 18, 1895.
† *Ibid.*, **8**, pt. ii., p. 213, 1899.
‡ *Ibid.*, **10**, pt. i., p. 155, 1925.
§ *Beob.-Ergeb. der König. Sternwarte zu Berlin*, No. 11, p. 19, 1902.

Brown adopts for the ratio E/M the value 81·530, obtained by Hinks from the observations of Eros.

We may write

$$-\mathrm{P} = k \,.\, \pi,$$

where $\pi$ is the solar parallax in seconds of arc and $k$ is a constant. Corresponding values of $-$ P and $\pi$ on Brown's theory are 125″·154 and 8″·80549.

The value of $\pi$ corresponding to $-$ P = 125″·023 is then

$$\pi = 8''{\cdot}796 \pm 0''{\cdot}002_3.$$

The probable error is based upon the total weight of the determination of P and a probable error for unit weight of ± 1″·25. It must be remembered, however, that the determination of P has been based upon an adopted value of the semi-diameter of the Moon, and that owing to the correlation between the parallactic inequality and the semi-diameter, a change in the value of the semi-diameter used will affect the derived value of the parallactic inequality and therefore of the solar parallax. A change in the semi-diameter of 0″·04 will lead to a change in the value of P of about 0″·048 and in the derived value of the solar parallax of 0″·0034.

The probable error of the adopted semi-diameter, which is that obtained in the discussion of the Cape occultations from 1880 to 1892, is ± 0″·04. To allow for this we may increase the probable error of the principal coefficient in the parallactic inequality from ± 0″·033 to ± 0″·058, and we may therefore adopt as the final value of the solar parallax resulting from the occultations used,

$$8''{\cdot}796 \pm 0''{\cdot}004,$$

the assigned probable error now giving a true measure of the uncertainty of the result. As all bright-limb phenomena, except for very bright stars, have been excluded, no systematic errors arising from disappearances at the bright limb being recorded too soon, and reappearances being recorded too late, will have entered.

The value of the solar parallax derived from the discussion of the Cape occultations from 1880 to 1922 was

$$8''{\cdot}804 \pm 0''{\cdot}005.$$

As the weighted mean value of the two determinations we may adopt

$$8''{\cdot}799 \pm 0''{\cdot}003.$$

From the Greenwich and Washington meridian observations of the Moon, Newcomb derived the parallactic inequality and thence a solar parallax of 8″·802 ± 0″·008. Incorporating the results of Battermann's discussion of occultations and Franz's observations of Mösting A, he adopted as the value of the solar parallax derived from the parallactic inequality, 8″·794 ± 0″·008.* The values are in good agreement with the above mean value.

* *Astronomical Constants*, pp. 151–2.

The derivation of the solar parallax from the parallactic inequality has not received the attention which it deserves. This is because the determination of the parallactic inequality from meridian observations is bound up with errors of a systematic nature arising from the fact that the observations are made on one limb of the Moon when the inequality is positive, and on the other limb when it is negative, and the effects due to irradiation vary according as the observation is secured in daylight, during twilight, or at night. Observations of occultations are not liable to be affected by systematic errors, except possibly observations of disappearance or of reappearance of faint stars at the bright limb. Newcomb realised the power of the method and the great superiority of occultations to meridian observations for the purpose. It was apparently his intention to include the determination of the parallactic inequality along with those of the other constants in his occultation memoir, but the parallactic correction term was dropped from his normal equations without any explanation. The agreement of the value derived from the rediscussion of occultations in Newcomb's memoir with that derived from the extensive series of Cape occultations lends strong support to the view that the method is free from any serious systematic errors.

An extensive series of occultation observations, reduced with positions of the Moon computed from Brown's theory, should give a particularly good determination of the parallactic inequality. It would be free from small systematic errors which may have entered through the omission of short-period terms with small coefficients and periods approximating to that of the Moon's rotation period. The present discussion, in conjunction with the discussion of the Cape occultations, has closely confirmed the fundamental constants of Brown's theory, so that it would only be necessary to include terms depending upon the longitude, the shape of the limb, the parallactic inequality, and the semi-diameter.

### *The Longitude of the Moon.*

In his occultation memoir, Newcomb derived the longitude corrections by neglecting the corrections to the other elements, and these corrections were then used to eliminate the longitude terms from the equations of condition before forming the normal equations. When, as happens in some years, there are several nights, each with a large number of occultations, the effect of the neglected terms can become appreciable and amount to as much as $0''{\cdot}5$. In the present discussion the normal equations were formed in the first instance with the longitude correction for each year as an unknown; preliminary values of the longitude corrections were used to derive the remaining corrections, and after several approximations to these in the form $a + bT$ the semi-definitive values for these corrections were substituted in the normal equations for the longitudes to derive definitive values of the longitude corrections. Smoothed longitude corrections, based on these, were employed for the derivation of the definitive corrections to the other elements.

Corrections were applied in the reductions to reduce to the mean longitude of Brown's Tables. The longitude corrections finally derived are given in Tables III. and IV. Table III. contains longitude corrections for the period 1672–1820; the material for these years is scanty, and it has been necessary to group the observations, with several years in a group. In Table IV. are given the corrections for each year for the period 1820–1908. The weights vary considerably from year to year, and are indicated by the subscript figures.

TABLE III.

*Longitude Corrections*, 1672–1820 (*Obs.-Tab.*).

| Mean Date. | Correction. | Weight. | Mean Date. | Correction. | Weight. |
|---|---|---|---|---|---|
| 1681 | $-1''.65$ | 11 | 1792·6 | $+0''.40$ | 16 |
| 1710 | −0·45 | 25 | 1800·4 | −0·31 | 18 |
| 1727 | +0·01 | 7 | 1803·2 | −0·87 | 17 |
| 1738 | +1·00 | 14 | 1810·0 | −0·59 | 16 |
| 1747 | +0·95 | 13 | 1811·5 | −1·28 | 12 |
| 1760·9 | +0·22 | 10 | 1812·5 | −0·17 | 13 |
| 1774·1 | +0·92 | 22 | 1813·5 | −1·34 | 16 |
| 1785·5 | +1·23 | 15 | | | |

TABLE IV.

*Longitude Corrections*, 1820–1908 (*Obs.-Tab.*).

| Year. | 1820. | 1830. | 1840. | 1850. | 1860. | 1870. | 1880. | 1890. | 1900. |
|---|---|---|---|---|---|---|---|---|---|
| 0·5 | $-0''.00$ | $-0''.61_{23}$ | $-1''.15_{22}$ | $+1''.63_{38}$ | $+3''.44_{24}$ | $+1''.11_{13}$ | $-2''.70_{43}$ | $-2''.64_{23}$ | $-1''.46_{3}$ |
| 1·5 | $-0''.20_{26}$ | $-1''.32_{31}$ | $-0''.25_{14}$ | $+1''.95_{22}$ | $+3''.21_{10}$ | $-0''.39_{19}$ | $-2''.20_{20}$ | $-2''.76_{30}$ | $-1''.30_{3}$ |
| 2·5 | $+0''.28_{13}$ | $-0''.47_{10}$ | $-1''.15_{6}$ | $+1''.83_{14}$ | $+2''.60_{13}$ | $-0''.82_{23}$ | $-2''.51_{17}$ | $-2''.84_{25}$ | $+0''.05_{21}$ |
| 3·5 | $-0''.66_{4}$ | $+0''.39_{1}$ | $+0''.77_{9}$ | $+2''.10_{16}$ | $+2''.11_{27}$ | $-1''.20_{28}$ | $-3''.17_{16}$ | $-2''.53_{25}$ | $+0''.37_{7}$ |
| 4·5 | $-0''.70_{4}$ | $-1''.65_{9}$ | $-0''.35_{4}$ | $+2''.36_{30}$ | $+2''.58_{20}$ | $-1''.72_{18}$ | $-2''.46_{59}$ | $-3''.11_{55}$ | $+0''.81_{4}$ |
| 5·5 | $-1''.06_{7}$ | $-1''.41_{8}$ | $-0''.15_{15}$ | $+2''.53_{12}$ | $+2''.23_{9}$ | $-2''.04_{16}$ | $-2''.38_{130}$ | $-2''.72_{212}$ | $+1''.31_{4}$ |
| 6·5 | $-0''.11_{5}$ | $-1''.35_{7}$ | $+1''.28_{6}$ | $+2''.15_{9}$ | $+2''.98_{13}$ | $-2''.07_{67}$ | $-2''.52_{35}$ | $-2''.20_{202}$ | $+2''.28_{5}$ |
| 7·5 | $-1''.26_{8}$ | $-1''.25_{8}$ | $+0''.51_{14}$ | $+2''.68_{15}$ | $+1''.94_{10}$ | $-3''.09_{15}$ | $-2''.24_{46}$ | $-3''.20_{30}$ | $+2''.32_{3}$ |
| 8·5 | $-0''.32_{3}$ | $-1''.42_{9}$ | $+1''.89_{14}$ | $+3''.19_{15}$ | $+1''.41_{22}$ | $-1''.96_{21}$ | $-2''.85_{15}$ | $-2''.78_{42}$ | $+2''.96_{1}$ |
| 9·5 | $-1''.49_{10}$ | $-0''.99_{21}$ | $+0''.70_{27}$ | $+2''.79_{13}$ | $+2''.07_{17}$ | $-1''.89_{28}$ | $-2''.67_{14}$ | $-2''.53_{24}$ | |

These longitude corrections were plotted and a smooth curve was drawn to represent the plotted points as closely as possible. The corrections read from this curve were adopted as the definitive longitude corrections in the subsequent work.

The corrections in the above tables are corrections to the longitude according to Brown's Tables, and therefore include the great empirical

term. In order to reduce to pure theory, this term must be removed. The secular acceleration should also be corrected to the value based on ancient observations of eclipses and other phenomena. The most reliable determination is that by de Sitter,* but the determination by Fotheringham † has been adopted in order that the results from the occultations may be immediately comparable with the corresponding results from the Greenwich meridian observations as given by Brown in his paper on the rotation of the Earth.‡

The following reduction from Brown's theory to pure theory has therefore been adopted :—

$$-2''\cdot71 + 2''\cdot02T + 4''\cdot79T^2 - 10''\cdot71 \sin(140^\circ T + 100^\circ\cdot7)$$

where T here denotes the time in centuries elapsed since 1800·0.

The results are summarised in Table V. The occultation values have been read off from a smooth curve ; in view of the very unequal weight of the occultation results in different years, this was the only course possible. The meridian results, on the other hand, are the unsmoothed observed values. From 1750 to 1850 Brown gives the meridian results for successive Cowell periods ; from 1850 onwards they are given year by year.

TABLE V.

*Longitude Differences : Theory* minus *Observation.*

| Date. | Longitude Correction. | Reduction to Theory. | Th.—Occ. | Th.—G. | Occ.—G. |
|---|---|---|---|---|---|
| | ″ | ″ | ″ | ″ | ″ |
| 1681 | −1·65 | +11·43 | +13·08 | .. | .. |
| 1710 | −0·48 | + 3·92 | + 4·40 | .. | .. |
| 1720 | −0·07 | + 0·84 | + 0·91 | .. | .. |
| 1727 | +0·28 | + 1·35 | − 1·63 | .. | .. |
| 1738 | +0·74 | − 4·70 | − 5·44 | .. | .. |
| 1744 | +0·78 | − 6·41 | − 7·19 | .. | .. |
| 1747 | +0·75 | − 7·22 | − 7·97 | .. | .. |
| 1751·3 | +0·68 | − 8·32 | − 9·00 | − 5·85 | +3·15 |
| 2·4 | +0·66 | − 8·59 | − 9·25 | − 9·22 | +0·03 |
| 3·6 | +0·64 | − 8·85 | − 9·49 | − 9·39 | +0·10 |
| 4·7 | +0·63 | − 9·12 | − 9·75 | − 9·26 | +0·49 |
| 5·8 | +0·62 | − 9·39 | −10·01 | −10·31 | −0·30 |
| 7·0 | +0·62 | − 9·65 | −10·27 | − 9·66 | +0·61 |
| 8·1 | +0·62 | − 9·90 | −10·52 | − 9·31 | +1·21 |
| 9·2 | +0·63 | −10·13 | −10·76 | − 8·46 | +2·30 |
| 1760·4 | +0·64 | −10·35 | −10·99 | − 9·08 | +1·91 |

* On the secular accelerations and the fluctuations of the longitudes of the Moon, the Sun, Mercury, and Venus (*B.A.N.*, **4**, No. 124, 1927).

† *M.N.R.A.S.*, **81**, 125, 1920.

‡ *Yale University Observatory Transactions*, **3**, pt. vi., 1926.

Table V.—*continued.*

| Date. | Longitude Correction. | Reduction to Theory. | Th.—Occ. | Th.—G. | Occ.—G. |
|---|---|---|---|---|---|
| 1761·5 | +0″·65 | −10″·57 | −11″·22 | − 9″·79 | +1″·43 |
| 2·6 | +0·66 | −10·80 | −11·46 | −10·11 | +1·35 |
| 3·8 | +0·70 | −11·02 | −11·72 | − 9·22 | +2·50 |
| 4·9 | +0·73 | −11·22 | −11·95 | −12·33 | −0·38 |
| 6·0 | +0·76 | −11·41 | −12·17 | − 9·92 | +2·25 |
| 7·2 | +0·79 | −11·60 | −12·39 | −11·00 | +1·39 |
| 8·3 | +0·81 | −11·78 | −12·59 | −11·48 | +1·11 |
| 9·4 | +0·83 | −11·95 | −12·78 | −12·95 | −0·17 |
| 1770·6 | +0·86 | −12·12 | −12·98 | −14·02 | −1·04 |
| 1·7 | +0·90 | −12·27 | −13·17 | −12·47 | +0·70 |
| 2·8 | +0·94 | −12·42 | −13·36 | −11·62 | +1·74 |
| 4·0 | +0·97 | −12·56 | −13·53 | −12·76 | +0·77 |
| 5·1 | +0·99 | −12·68 | −13·67 | −13·69 | −0·02 |
| 6·2 | +1·00 | −12·80 | −13·80 | −14·33 | −0·53 |
| 7·4 | +1·01 | −12·92 | −13·93 | −12·93 | +1·00 |
| 8·5 | +1·01 | −13·02 | −14·03 | −13·43 | +0·60 |
| 9·6 | +1·00 | −13·11 | −14·11 | −15·93 | −1·82 |
| 1780·8 | +0·99 | −13·20 | −14·19 | −14·42 | −0·23 |
| 1·9 | +0·98 | −13·28 | −14·26 | −13·29 | +0·97 |
| 3·0 | +0·97 | −13·34 | −14·31 | −15·34 | −1·03 |
| 4·2 | +0·96 | −13·40 | −14·36 | −15·49 | −1·13 |
| 5·3 | +0·94 | −13·46 | −14·40 | −14·73 | −0·33 |
| 6·4 | +0·90 | −13·50 | −14·40 | −14·69 | −0·29 |
| 7·6 | +0·84 | −13·53 | −14·37 | −14·41 | −0·04 |
| 8·7 | +0·78 | −13·55 | −14·33 | −16·54 | −2·21 |
| 9·8 | +0·71 | −13·55 | −14·26 | −16·05 | −1·79 |
| 1791·0 | +0·63 | −13·55 | −14·18 | −16·35 | −2·17 |
| 2·1 | +0·55 | −13·55 | −14·10 | −15·94 | −1·84 |
| 3·2 | +0·42 | −13·54 | −13·96 | −16·22 | −2·26 |
| 4·4 | +0·30 | −13·51 | −13·81 | −16·09 | −2·28 |
| 5·5 | +0·17 | −13·47 | −13·64 | −14·86 | −1·22 |
| 6·6 | +0·02 | −13·43 | −13·45 | −12·02 | +1·43 |
| 7·8 | −0·09 | −13·37 | −13·28 | −12·99 | +0·29 |
| 8·9 | −0·19 | −13·30 | −13·11 | −14·03 | −0·92 |
| 1800·0 | −0·28 | −13·23 | −12·95 | −12·95 | 0·00 |
| 1·2 | −0·38 | −13·15 | −12·77 | −13·06 | −0·29 |
| 2·3 | −0·45 | −13·05 | −12·60 | −12·28 | +0·32 |
| 3·4 | −0·52 | −12·95 | −12·43 | −12·27 | +0·16 |

TABLE V.—*continued.*

| Date. | Longitude Correction. | Reduction to Theory. | Th.−Occ. | Th.−G. | Occ.−G. |
|---|---|---|---|---|---|
| 1804·6 | −0″·60 | −12″·84 | −12″·24 | −11″·26 | +0″·98 |
| 5·7 | −0·63 | −12·72 | −12·09 | −10·73 | +1·36 |
| 6·8 | −0·68 | −12·60 | −11·92 | −11·69 | +0·23 |
| 8·0 | −0·73 | −12·46 | −11·73 | −11·36 | +0·37 |
| 9·1 | −0·76 | −12·31 | −11·55 | −11·92 | −0·37 |
| 1810·2 | −0·78 | −12·15 | −11·37 | −11·67 | −0·30 |
| 1·4 | −0·78 | −11·99 | −11·21 | − 9·79 | +1·42 |
| 2·5 | −0·76 | −11·82 | −11·06 | − 9·51 | +0·55 |
| 3·6 | −0·71 | −11·64 | −10·93 | − 9·24 | +1·69 |
| 4·8 | −0·65 | −11·45 | −10·80 | − 7·46 | +3·34 |
| 5·9 | −0·60 | −11·25 | −10·65 | − 8·96 | +1·69 |
| 7·0 | −0·53 | −11·05 | −10·52 | − 8·25 | +2·27 |
| 8·2 | −0·43 | −10·83 | −10·40 | − 8·23 | +2·17 |
| 9·3 | −0·36 | −10·61 | −10·25 | − 6·60 | +3·65 |
| 1820·4 | −0·30 | −10·39 | −10·09 | − 7·07 | +3·02 |
| 1·6 | −0·31 | −10·15 | − 9·84 | − 7·43 | +2·41 |
| 2·7 | −0·34 | − 9·91 | − 9·57 | − 7·49 | +2·08 |
| 3·8 | −0·43 | − 9·66 | − 9·23 | − 8·03 | +1·20 |
| 5·0 | −0·53 | − 9·39 | − 8·86 | − 8·18 | +0·68 |
| 6·1 | −0·61 | − 9·12 | − 8·51 | − 7·72 | +0·79 |
| 7·2 | −0·72 | − 8·86 | − 8·14 | − 7·54 | +0·60 |
| 8·4 | −0·85 | − 8·57 | − 7·72 | − 6·87 | +0·85 |
| 9·5 | −0·94 | − 8·29 | − 7·35 | − 7·58 | −0·23 |
| 1830·6 | −1·02 | − 8·00 | − 6·98 | − 6·50 | +0·48 |
| 1·8 | −1·14 | − 7·70 | − 6·56 | − 5·50 | +1·06 |
| 2·9 | −1·24 | − 7·40 | − 6·16 | − 4·59 | +1·57 |
| 4·0 | −1·34 | − 7·10 | − 5·76 | − 4·88 | +0·88 |
| 5·2 | −1·43 | − 6·78 | − 5·35 | − 4·56 | +0·79 |
| 6·3 | −1·41 | − 6·45 | − 5·04 | − 4·14 | +0·90 |
| 7·4 | −1·35 | − 6·12 | − 4·77 | − 3·81 | +0·96 |
| 8·6 | −1·27 | − 5·78 | − 4·51 | − 3·38 | +1·13 |
| 9·7 | −1·09 | − 5·45 | − 4·36 | − 3·94 | +0·42 |
| 1840·8 | −0·81 | − 5·12 | − 4·31 | − 3·20 | +1·11 |
| 2·0 | −0·49 | − 4·77 | − 4·28 | − 3·55 | +0·73 |
| 3·1 | −0·20 | − 4·42 | − 4·22 | − 3·80 | +0·42 |
| 4·2 | +0·10 | − 4·08 | − 4·18 | − 3·75 | +0·43 |
| 5·4 | +0·41 | − 3·72 | − 4·13 | − 4·29 | −0·16 |
| 6·5 | +0·77 | − 3·35 | − 4·12 | − 3·04 | +1·08 |

TABLE V.—*continued.*

| Date. | Longitude Correction. | Reduction to Theory. | Th.−Occ. | Th.−G. | Occ.−G. |
|---|---|---|---|---|---|
| | ″ | ″ | ″ | ″ | ″ |
| 1847·6 | +1·05 | − 3·00 | − 4·05 | − 3·37 | +0·68 |
| 8·8 | +1·31 | − 2·63 | − 3·94 | − 4·11 | −0·17 |
| 9·9 | +1·55 | − 2·26 | − 3·81 | − 3·40 | +0·41 |
| 1850·5 | +1·66 | − 2·07 | − 3·73 | − 3·26 | +0·47 |
| 1 | +1·82 | − 1·74 | − 3·56 | − 3·58 | −0·02 |
| 2 | +1·96 | − 1·41 | − 3·37 | − 3·83 | −0·46 |
| 3 | +2·14 | − 1·08 | − 3·22 | − 3·43 | −0·21 |
| 4 | +2·31 | − 0·75 | − 3·06 | − 3·46 | −0·40 |
| 5 | +2·43 | − 0·42 | − 2·85 | − 3·37 | −0·52 |
| 6 | +2·56 | − 0·08 | − 2·64 | − 3·16 | −0·52 |
| 7 | +2·72 | + 0·26 | − 2·46 | − 2·76 | −0·30 |
| 8 | +2·93 | + 0·60 | − 2·33 | − 2·35 | −0·02 |
| 9 | +3·09 | + 0·94 | − 2·15 | − 2·05 | +0·10 |
| 1860·5 | +3·16 | + 1·27 | − 1·89 | − 1·82 | +0·07 |
| 1 | +3·11 | + 1·66 | − 1·45 | − 1·61 | −0·16 |
| 2 | +2·97 | + 1·95 | − 1·02 | − 1·46 | −0·44 |
| 3 | +2·76 | + 2·29 | − 0·47 | − 1·46 | −0·99 |
| 4 | +2·59 | + 2·63 | + 0·04 | − 0·80 | −0·84 |
| 5 | +2·48 | + 2·97 | + 0·49 | 0·00 | −0·49 |
| 6 | +2·38 | + 3·31 | + 0·93 | + 0·82 | −0·11 |
| 7 | +2·24 | + 3·65 | + 1·41 | + 1·62 | +0·21 |
| 8 | +2·00 | + 3·98 | + 1·98 | + 2·45 | +0·47 |
| 9 | +1·65 | + 4·32 | + 2·67 | + 3·10 | +0·43 |
| 1870·5 | +1·18 | + 4·65 | + 3·47 | + 3·75 | +0·28 |
| 1 | −0·10 | + 4·98 | + 5·08 | + 4·81 | −0·27 |
| 2 | −0·85 | + 5·32 | + 6·17 | + 5·33 | −0·84 |
| 3 | −1·32 | + 5·65 | + 6·97 | + 6·52 | −0·45 |
| 4 | −1·66 | + 5·98 | + 7·64 | + 7·71 | +0·07 |
| 5 | −1·91 | + 6·30 | + 8·21 | + 8·55 | +0·34 |
| 6 | −2·05 | + 6·63 | + 8·68 | + 8·80 | +0·12 |
| 7 | −2·15 | + 6·96 | + 9·11 | + 8·95 | −0·16 |
| 8 | −2·23 | + 7·28 | + 9·51 | + 9·56 | +0·05 |
| 9 | −2·30 | + 7·60 | + 9·90 | + 9·65 | −0·25 |
| 1880·5 | −2·35 | + 7·92 | +10·27 | + 9·91 | −0·36 |
| 1 | −2·40 | + 8·23 | +10·63 | +10·06 | −0·57 |
| 2 | −2·44 | + 8·54 | +10·98 | +10·43 | −0·55 |
| 3 | −2·48 | + 8·85 | +11·33 | +10·85 | −0·48 |
| 4 | −2·52 | + 9·16 | +11·68 | +11·36 | −0·32 |

TABLE V.—*continued.*

| Date. | Longitude Correction. | Reduction to Theory. | Th.—Occ. | Th.—G. | Occ.—G. |
|---|---|---|---|---|---|
| | ″ | ″ | ″ | ″ | ″ |
| 1885 | −2·55 | + 9·46 | +12·01 | +12·14 | +0·13 |
| 6 | −2·58 | + 9·76 | +12·34 | +12·63 | +0·29 |
| 7 | −2·61 | +10·06 | +12·67 | +12·59 | −0·08 |
| 8 | −2·64 | +10·36 | +13·00 | +12·99 | −0·01 |
| 9 | −2·66 | +10·65 | +13·31 | +13·04 | −0·27 |
| 1890·5 | −2·68 | +10·93 | +13·61 | +13·20 | −0·41 |
| 1 | −2·70 | +11·21 | +13·91 | +14·05 | +0·14 |
| 2 | −2·71 | +11·48 | +14·19 | +14·89 | +0·70 |
| 3 | −2·72 | +11·75 | +14·47 | +15·24 | +0·77 |
| 4 | −2·73 | +12·03 | +14·76 | +15·35 | +0·59 |
| 5 | −2·72 | +12·30 | +15·02 | +15·44 | +0·42 |
| 6 | −2·70 | +12·56 | +15·26 | +15·73 | +0·47 |
| 7 | −2·66 | +12·81 | +15·47 | +15·76 | +0·29 |
| 8 | −2·54 | +13·07 | +15·61 | +15·36 | −0·25 |
| 9 | −2·30 | +13·32 | +15·62 | +15·21 | −0·42 |
| 1900·5 | −1·76 | +13·56 | +15·32 | +15·03 | −0·29 |
| 1 | −0·98 | +13·80 | +14·78 | +14·72 | −0·06 |
| 2 | −0·25 | +14·03 | +14·28 | +14·30 | +0·02 |
| 3 | +0·36 | +14·26 | +13·90 | +13·85 | −0·05 |
| 4 | +0·94 | +14·49 | +13·55 | +13·50 | −0·05 |
| 5 | +1·54 | +14·71 | +13·17 | +12·98 | −0·19 |
| 6 | +2·13 | +14·92 | +12·79 | +12·58 | −0·21 |
| 7 | +2·60 | +15·13 | +12·53 | +12·48 | −0·05 |
| 8 | +2·96 | +15·33 | +12·37 | +12·32 | −0·05 |

The differences between the longitude corrections derived in the present investigation and those derived by Newcomb are on the whole small, in a few exceptional instances being as large as 1″. The systematic differences between Newcomb's occultation values and the Greenwich meridian values therefore persist. They are most pronounced in the period from 1788 to 1830. The differences, occultations minus Greenwich, increase from − 2″ in 1788 to + 3″ in 1814 and to nearly + 4″ in 1819, and then decrease to zero in 1829. The Greenwich errors of the Sun's longitude for this period are very erratic and run in a closely analogous manner; the difference, theory minus Greenwich, increases from − 2″ about 1785 to + 4″ about 1814, and decreases to zero about 1830. During this period the Greenwich meridian observations are therefore apparently affected by variable systematic errors and the occultation results are to be preferred. From 1830 onwards the differences between the occultations and the Greenwich meridian observations are much smaller, rarely amounting to 1″. There are

several periods, however, in which the differences appear to indicate small systematic errors in one or both series. During the last thirty years covered by the table, the differences, with one or two exceptions, are satisfactorily small.

Brown * has called attention to an apparent correlation between the minor fluctuations of the Moon and the differences between the longitudes given by the occultations and meridian observations. With the revised results from the occultations, the correlation becomes less marked, especially during the later years covered by the comparison. The apparent correlation during the earlier period appears to be due to the systematic errors of the Greenwich observations: it has been shown that the differences between the occultations and the Greenwich observations of the Moon run closely parallel to the differences between the Sun's tabular longitude and the Greenwich observed longitude. In other words, the Greenwich observations of both Sun and Moon have been affected by similar errors. But during the same period the minor fluctuations of the Moon happen to run a somewhat similar course, increasing from about 1790 to 1820 and thereafter gradually decreasing.

De Sitter † represents the errors of the Moon's longitude by a series of straight lines, with changes of slope at 1667, 1758, 1784, 1864, 1876, 1897, and 1917. Such a series of straight lines does not represent the occultation results in detail; thus the change at 1758 occurs according to the occultations about 1745, and between 1784 and 1864 there are changes at 1820, at 1838 (particularly well marked), and at 1848; subsequent changes occur about 1860, 1870, 1875, and 1899. The differences in some of these dates are due to the differences between the occultation and meridian results, as de Sitter used occultations up to 1835 and Greenwich meridian observations after 1835.

In his investigation of the fluctuations of the longitudes of the Moon, the Sun, Mercury, and Venus, de Sitter † introduced into the expressions for the longitude residuals of the Sun, Mercury, and Venus a term of the form $Qn/n' \cdot B'$, where $n$, $n'$ denote the mean motions of the Sun or planet and of the Moon respectively, and $B'$ denotes the total fluctuation in the longitude of the Moon. The values of Q, derived separately from the Sun, Venus, and Mercury, were reasonably concordant, with a mean value of about 1·25. The longitudes of the Moon used by de Sitter were partly derived from occultations and partly from meridian observations.

Using the occultation results alone, I have repeated de Sitter's discussion, with some slight modifications, and derive the following values of Q:—

| | |
|---|---|
| From the Sun | $+ 1{\cdot}33 \pm 0{\cdot}08$ |
| ,, Venus | $+ 1{\cdot}32 \pm 0{\cdot}07$ |
| ,, Mercury | $+ 1{\cdot}15 \pm 0{\cdot}04$. |

The separate values agree with de Sitter's within their probable errors.

* *Yale University Observatory Transactions*, **3**, pt. vi., p. 227, 1926.
† *B.A.N.*, **4**, No. 124, 1927.

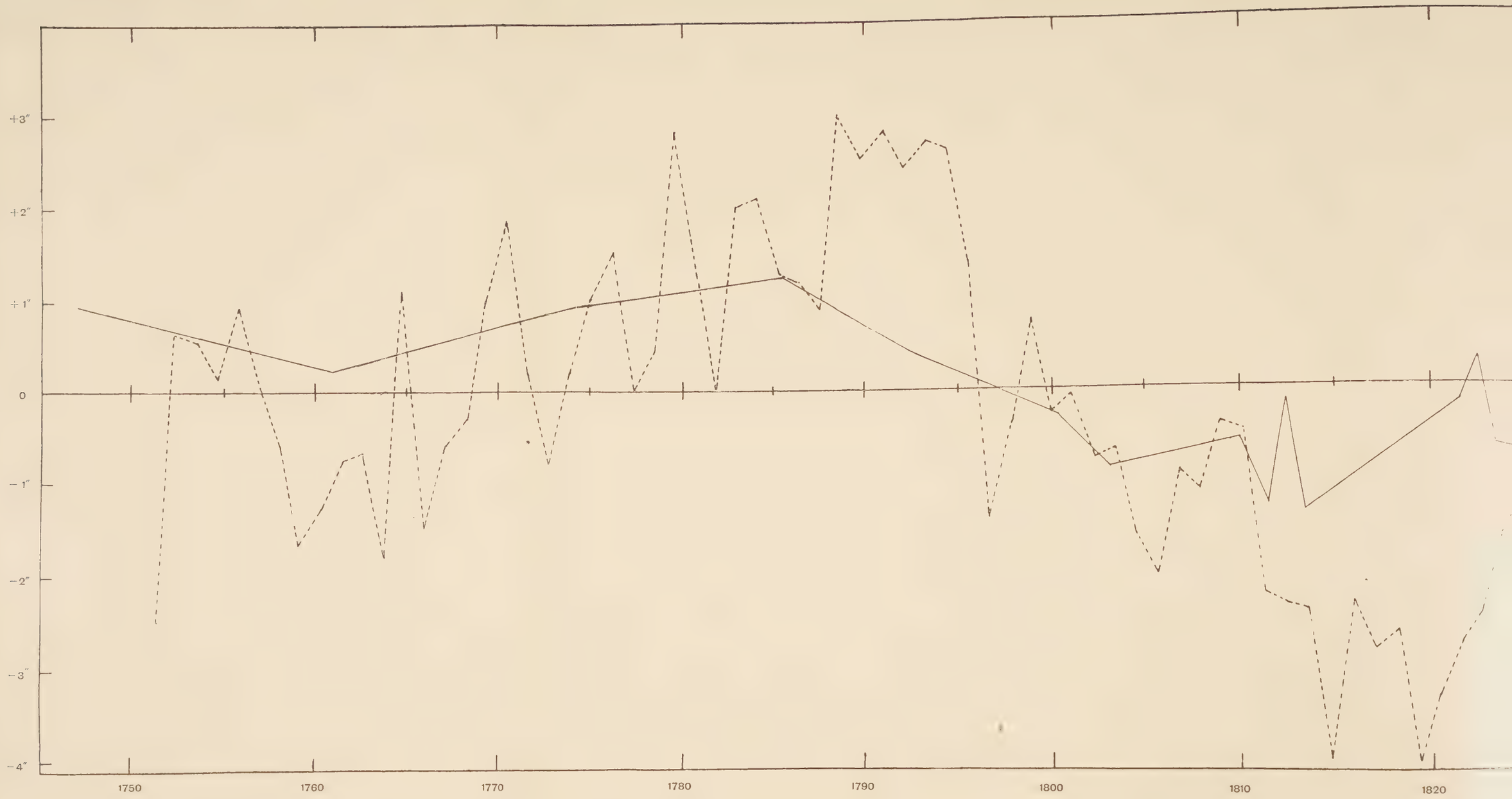
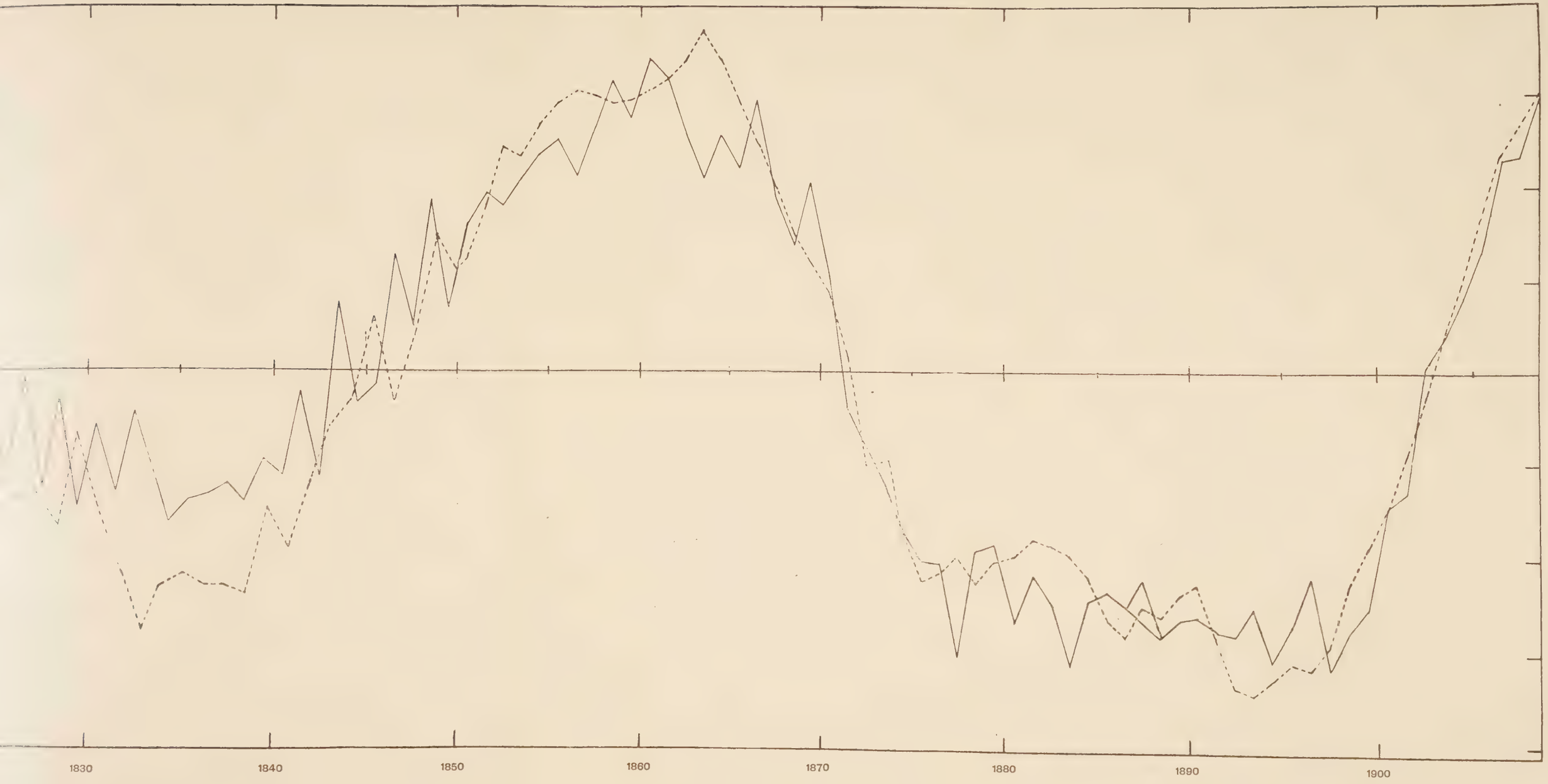



Fig. 1.—Errors in Longitude of Moon from Brown's Tables (Obs.–Tab.).
----- Greenwich Meridian Observations.
—— Occultation Observations.


1830
1840
1850
1860
1870
1880
1890
1900

The occultation results may therefore be used alone as a basis of any discussion of this nature.

In fig. 1 are shown the errors in longitude of the Moon from Brown's Tables, as derived from the occultations and from the Greenwich meridian observations. In fig. 2 is shown the curve of the errors when the longitude of the Moon is that derived from pure theory; the occultation longitudes are the observed unsmoothed values. In fig. 3 are shown the longitude differences between the occultations and the Greenwich meridian observations. The occultation results have been smoothed and the curve is only intended to illustrate the systematic nature of the differences during certain periods.

## *Summary.*

1. The observations of occultations collected in Newcomb's "Researches on the Motion of the Moon," Part II., have been reduced to the basis of Brown's Tables, and corrections to various constants have been derived.

2. The mean longitudes of the Moon's perigee and node, adopted in Brown's Tables and based on the Greenwich meridian observations, are closely confirmed. The mean motions of both perigee and node derived from the occultations agree within one second of arc per century with the mean motions given by the meridian observations.

3. The coefficient of the principal term in the latitude is $18461''\cdot39 \pm 0''\cdot05$, in almost exact agreement with Brown's adopted value.

4. A mean correction to the Moon's tabular latitude of $-0''\cdot49 \pm 0''\cdot03$ is derived. The corrections from emersion and immersion observations separately are respectively $-0''\cdot28$ and $-0''\cdot61$. These corrections are interpreted as due to a depression of the N.E. to E. portion of the Moon's limb and a smaller depression of the N.W. to W. portion as compared with the corresponding southern portions of the limb.

5. A correction to Newcomb's value of the secular change of the obliquity of the ecliptic of amount $-0''\cdot37 \pm 0''\cdot14$ per century is indicated by the occultations. Incorporating other material, a mean correction of $-0''\cdot30$ is adopted. This is attributed to error in Newcomb's adopted mass of Venus. A reciprocal of the mass of Venus, of amount $402{,}910 \pm 1270$, is indicated.

6. The ellipticity of the Earth's figure required to suit the observed mean motions of the perigee and node is $1/293\cdot9$, if Brown's figure of the Moon terms are accepted. This value does not appear to suit the $\sin\alpha$ term in the latitude derived from the occultations, but other errors enter into the latter and render its interpretation somewhat uncertain. So large a value of the ellipticity does not accord with the observed constant of precession. The figure of the Moon terms corresponding to any ellipticity, and the related constants of the Moon's figure, are derived.

7. A correction to Newcomb's equinox, at epoch 1850, of amount

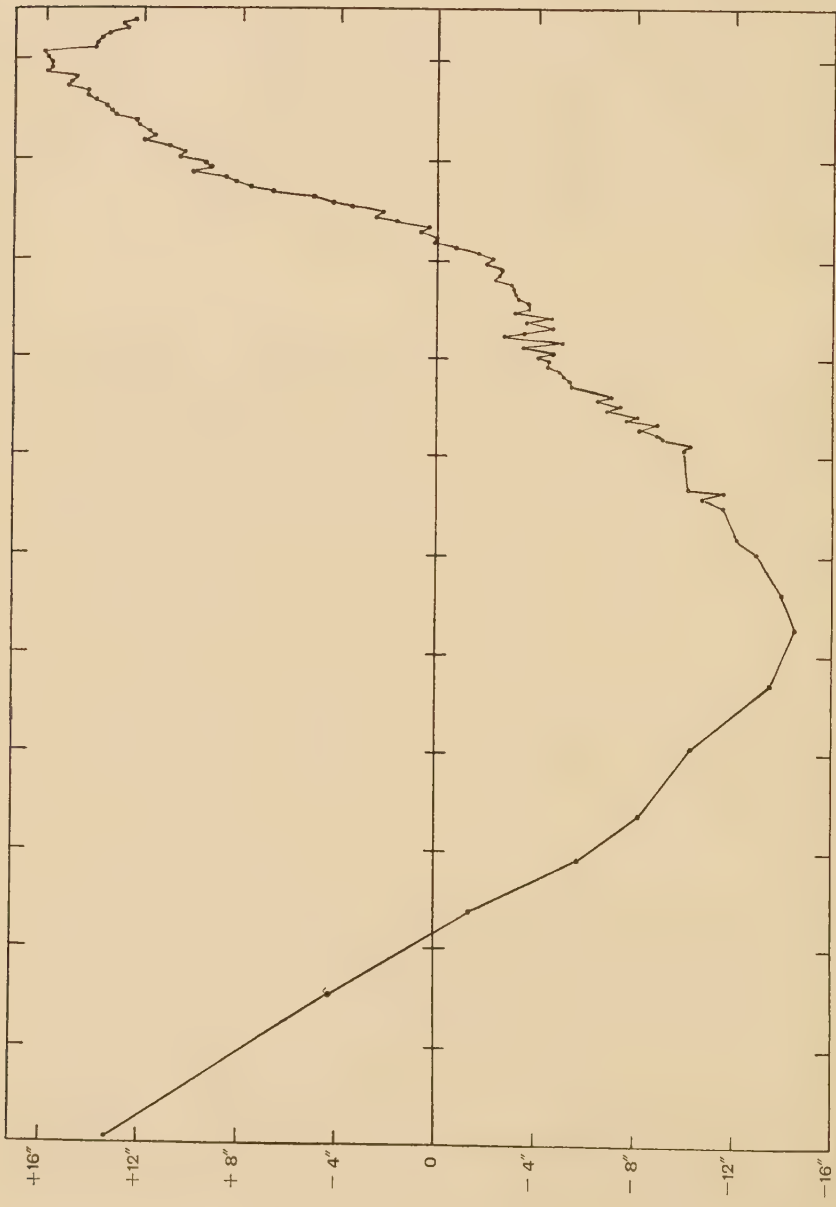



FIG. 2.—Longitude Differences: Theory *minus* Observations.

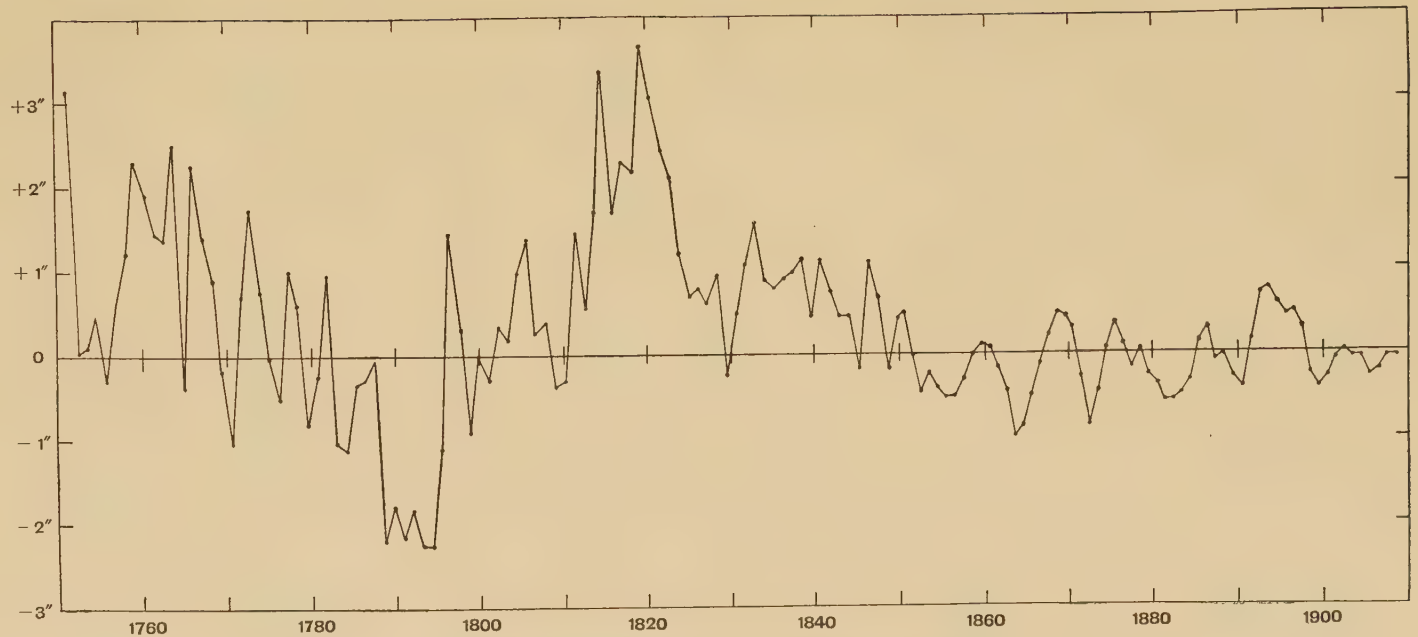



FIG. 3.—Longitude Differences, occultations *minus* Greenwich.

$-0^s{\cdot}047 \pm 0^s{\cdot}011$ is derived. The correction to the mean motion of the equinox, given by the occultations alone, is $+0^s{\cdot}036 \pm 0^s{\cdot}026$, but when modern meridian determinations are incorporated this is changed to $-0^s{\cdot}020 \pm 0^s{\cdot}015$.

8. Newcomb's declination system and proper motions in declination in the neighbourhood of the equator are closely confirmed.

9. The derived coefficient of the principal term in the parallactic inequality is $-125''{\cdot}023$, corresponding to a value for the solar parallax of $8''{\cdot}796 \pm 0''{\cdot}004$. The parallax derived from the Cape occultations, 1880-1922, was $8''{\cdot}804 \pm 0''{\cdot}005$. The mean value of the parallax is $8''{\cdot}799 \pm 0''{\cdot}003$.

10. The longitude corrections to Brown's Tables are given in Tables III. and IV. The corrections to a purely theoretical longitude, and comparison with the corrections derived from the Greenwich meridian observations, are given in Table V. The larger and more systematic differences between the occultations and meridian observations are attributable to errors in the latter.

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*The Colour Temperatures of Twenty-four Stars, suitable for use as Spectrophotometric Standards.* By. W. M. H. Greaves, C. Davidson, and E. Martin.

(*Communicated by the Astronomer Royal.*)

It has been shown in previous papers * how a photometric comparison between the continuous spectra of two stars yields the quantity $\frac{d \cdot \Delta m_\lambda}{d \cdot \frac{1}{\lambda}}$, where $\lambda$ denotes wave-length and $0{\cdot}4\Delta m_\lambda$ is the difference of the logarithms of the amounts of light falling into the spectra at wave-length $\lambda$. $\Delta m_\lambda$ is accordingly reckoned in magnitudes.

A direct application of Planck's formula leads to the equation

$$0{\cdot}922 \times \frac{d\Delta m_\lambda}{d \cdot \frac{1}{\lambda}} = \frac{c_2}{T_2}\left|\left(1 - e^{\frac{-c_2}{\lambda T_2}}\right)\right. - \frac{c_2}{T_1}\left|\left(1 - e^{\frac{-c_2}{\lambda T_1}}\right)\right.,$$

where $T_1$ and $T_2$ are the colour temperatures of the two stars, and $c_2 = 14{,}320$ (for $\lambda$ measured in microns) is the well-known constant in Planck's formula.

The quantity $\frac{d\Delta m_\lambda}{d \cdot \frac{1}{\lambda}}$ has been determined by a series of measures

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* *M.N.*, **86**, 33, 1925, and **87**, 352, 1927.

extending from $\lambda$ 4000 to $\lambda$ 6500, and on the right-hand side of the equation 0·5 has been substituted for $\lambda$. This gives a sufficiently near approximation.

Thus by means of a series of comparisons of pairs of stars the quantity

$$G = k + \frac{c_2}{T} \Big/ \left(1 - e^{\frac{-c_2}{0{\cdot}5T}}\right)$$

can be determined for each of any number of stars, $k$ being a constant which can be defined by choosing it such that G is zero for a given star or for the mean of a system. We shall aim at defining $k$ so that G is zero for the average star of type Ao.

G is called the "gradient." * The colour temperatures can be derived from the gradients if the colour temperature corresponding to zero gradient be known. To obtain a direct determination of this it would be necessary to obtain comparisons between stars and a terrestrial source of known colour temperature.

In stellar observations for the determination of gradient it is practically necessary to photograph each pair of stars at approximately equal altitudes so as to avoid introducing large corrections for selective atmospheric absorption. This condition restricts each comparison to a definite sidereal time and renders the straightforward method of comparing each star with one standard (*e.g.* Polaris) very irksome in a variable climate. In consequence 24 stars, fairly evenly distributed over the northern sky, have been selected to be used as standards. They have been inter-compared in pairs at equal altitudes so as to obtain the gradients of the whole set relative to a common zero.

The process of standardization was tedious, but the scheme being completed it is now possible to determine the gradient, relative to the zero of the system, of a star in any part of the sky, by comparison with convenient standards.

In this further work, in which considerable progress has already been made, it is proposed to compare each star with at least two standards.

The 24 standard stars are mostly of Type A, 6 of them being of

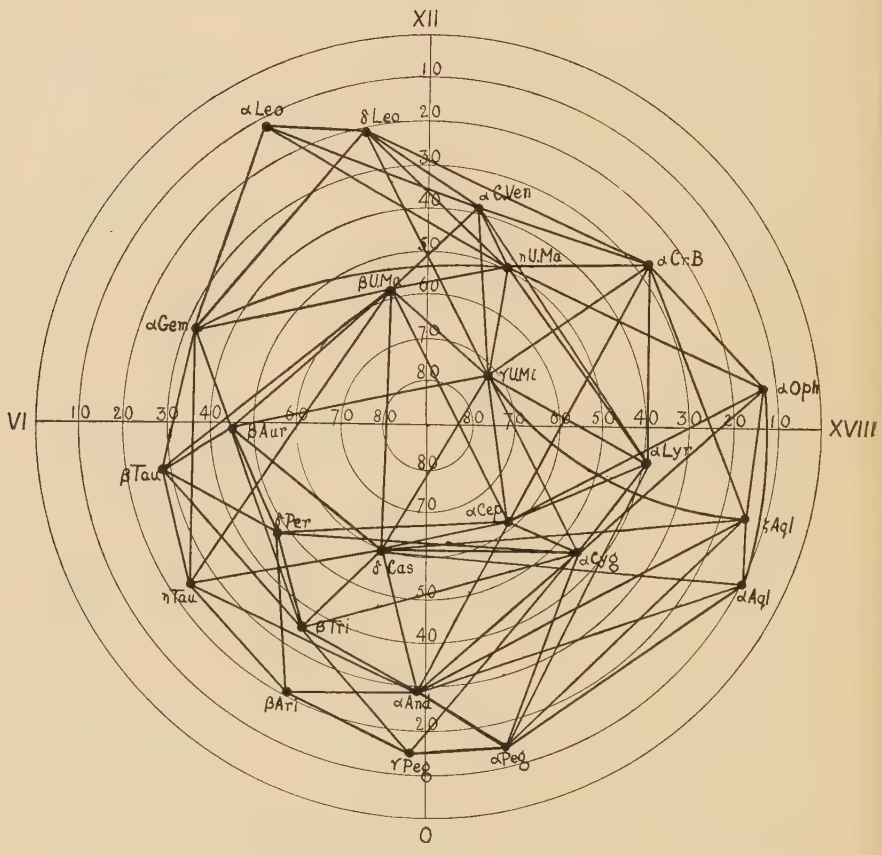
* The convention of sign adopted here for the gradients is the reverse of that used in previous papers. This change brings the notation into conformity with that of Sampson, the originator of the term "gradient." It has the further advantage of making the gradient of a star directly comparable with its colour index. The effective wave-lengths of the Harvard visual and photographic observations have been determined by Brill to be 0·529 $\mu$ and 0·425 $\mu$, and these give the following relation between gradient and colour index :—

$$G = 0{\cdot}922 \times \frac{CI}{\frac{1}{{\cdot}425} - \frac{1}{{\cdot}529}} = 2{\cdot}0 \times CI,$$

apart from the fact that the visual and photographic magnitudes used in the determination of colour index are observations in integrated light of an absorption spectrum, whereas the gradients are derived from measures of the continuous spectrum only.

Type B. It is desirable that standard stars should be of early type, as in work with low dispersion there is always a possibility of systematic error in the measurement of the more complicated spectra of the later types due to the difficulty of avoiding absorption lines and bands.

The present paper gives merely the results of the observations of the




Intercomparison of Standard Stars.

24 standard stars. Details will be given subsequently in a Greenwich publication. It is sufficient to mention here that since the publication of the preliminary results the prismatic camera has been reconstructed so as to give spectra in sharp focus throughout.* Further, the grating is no longer used for every exposure, the photometric scale being placed on each plate by two exposures with grating. A subsidiary investigation has demonstrated the absence of appreciable systematic error

* *Journal of Scientific Instruments*, **5**, 238, 1928.

depending on time of exposure so that the results are free from magnitude error.*

The scheme of observation employed in establishing the 24 standards is illustrated by the diagram, in which a line joining two stars denotes a series of comparisons usually 4 in number. In all, 325 comparisons between 85 pairs of standards were secured.

The difference of gradient $\Delta G = = -0{\cdot}922 \frac{d\Delta m_\lambda}{d \cdot \frac{1}{\lambda}}$ given by each comparison was corrected by the amount

$$A \cdot (\sec z_2 - \sec z_1),$$

where $z_1$ and $z_2$ are the zenith distances of the stars at the times of observation and the value 0·73 for A obtained previously was used. Since $z_1$ and $z_2$ were always approximately equal the correction was always small, its average value being about 0·03, and the effect of an admissible error in the adopted value of A was negligible, and could safely be included with the other accidental errors.

The gradients, and their probable errors, for the 24 stars, resulting from a least squares solution, are given in the following table, the constant $k$ having been adjusted as explained below.

TABLE I.

*Gradients of Standard Stars.*

| No. | Star's Name. | R.A. 1925·0. | Dec. 1925·0. | Vis. Mag. | Spectral Type. (Harvard.) | Gradient. | P.E. |
|---|---|---|---|---|---|---|---|
| | | h m | ° ′ | m | | | |
| 1 | $\alpha$ And. | 0 4·5 | +28 41 | 2·2 | A0*p* | −·062 | ±·010 |
| 2 | $\gamma$ Peg. | 0 9·4 | +14 46 | 2·9 | B2 | −·292 | ±·016 |
| 3 | $\delta$ Cas. | 1 20·9 | +59 51 | 2·8 | A5 | +·289 | ±·010 |
| 4 | $\beta$ Ari. | 1 50·5 | +20 27 | 2·7 | A5 | +·336 | ±·015 |
| 5 | $\beta$ Tri. | 2 5·1 | +34 38 | 3·1 | A5 | +·298 | ±·014 |
| 6 | $\delta$ Per. | 3 37·6 | +47 33 | 3·1 | B5 | −·073 | ±·013 |
| 7 | $\eta$ Tau. | 3 43·0 | +23 52 | 3·0 | B5*p* | +·012 | ±·014 |
| 8 | $\beta$ Tau. | 5 21·5 | +28 33 | 1·8 | B8 | −·112 | ±·015 |
| 9 | $\beta$ Aur. | 5 54·0 | +44 56 | 2·1 | A0*p* | +·108 | ±·012 |
| 10 | $\alpha$ Gem. (mean) | 7 29·8 | +32 3 | 1·6 | A0 | +·146 | ±·012 |

* This investigation consisted of a series of comparisons of a star with itself, the exposures being made with and without the grating. The ratio of the exposure times was about 15 : 1, and the spectrum of the short exposure was comparable in density with the first order image of the longer exposure. The mean gradient *long—short* was +0·012±·008 from 20 plates.

TABLE I.—*continued.*

| No. | Star's Name. | R.A. 1925·0. | Dec. 1925·0. | Vis. Mag. | Spectral Type. (Harvard.) | Gradient. | P.E. |
|---|---|---|---|---|---|---|---|
| | | h m | ° ′ | m | | | |
| 11 | $\alpha$ Leo. | 10 4·4 | +12 20 | 1·3 | B8 | −·043 | ±·014 |
| 12 | $\beta$ UMa. | 10 57·3 | +56 47 | 2·4 | A0 | +·030 | ±·010 |
| 13 | $\delta$ Leo. | 11 10·1 | +20 56 | 2·6 | A3 | +·330 | ±·012 |
| 14 | $\alpha$ CVn. (br.) | 12 52·5 | +38 43 | 2·9 | A0*p* | −·008 | ±·011 |
| 15 | $\eta$ UMa. | 13 44·6 | +49 41 | 1·9 | B3 | −·188 | ±·010 |
| 16 | $\gamma$ UMi. | 15 20·8 | +72 6 | 3·1 | A2 | +·228 | ±·009 |
| 17 | $\alpha$ CrB. | 15 31·5 | +26 58 | 2·3 | A0 | +·054 | ±·010 |
| 18 | $\alpha$ Oph. | 17 31·5 | +12 37 | 2·1 | A5 | +·358 | ±·012 |
| 19 | $\alpha$ Lyr. | 18 34·4 | +38 43 | 0·1 | A0 | +·052 | ±·010 |
| 20 | $\zeta$ Aql. | 19 2·0 | +13 45 | 3·0 | A0 | +·072 | ±·011 |
| 21 | $\alpha$ Aql. | 19 47·1 | + 8 40 | 0·9 | A5 | +·454 | ±·013 |
| 22 | $\alpha$ Cyg. | 20 38·9 | +45 1 | 1·3 | A2*p* | +·321 | ±·009 |
| 23 | $\alpha$ Cep. | 21 16·8 | +62 16 | 2·6 | A5 | +·501 | ±·009 |
| 24 | $\alpha$ Peg. | 23 1·0 | +14 48 | 2·6 | A0 | −·016 | ±·012 |

The probable error of a single comparison is ± 0·055 from a discussion of the residuals. The average probable error of the final gradients is ± ·012, corresponding to ± ·006 on the colour index scale. In subsequent comparisons of stars with the standards the gradients have been rounded off to 0·01 as in Table II. below.

In *Leiden Annals*, **14**, E. Hertzsprung has given a catalogue of the values of $\frac{c_2}{T}$ for 734 stars. These were obtained from all the colour data then available, and were calibrated on Wilsing's spectro-photometric results. In Table II. below is given a comparison between the gradients of Table I. and gradients for the same stars derived from Hertzsprung's data. The latter were converted into gradients by the relation

$$G = -1{\cdot}56 + \frac{c_2}{T} \Big/ \left(1 - e^{\frac{-2c_2}{T}}\right).$$

The constant − 1·56 is chosen so that for

$$G = 0, \qquad \frac{c_2}{T} = 1{\cdot}48,$$

this being the mean value for the 82 stars of Class A0 in Hertzsprung's catalogue. Actually the values used in the comparison were not those given in *Leiden Annals*, **14**, but revised values supplied in manuscript

by Prof. Hertzsprung, in which more recent colour data had been incorporated. Their mean, however, is the same as if *Leiden Annals* had been used.

TABLE II.

*Comparison with Hertzsprung's Data.*

| Star's Name. | Gradients. Greenwich. | Gradients. Hertzsprung. | Greenwich minus Hertzsprung. | Star's Name. | Gradients. Greenwich. | Gradients. Hertzsprung. | Greenwich minus Hertzsprung. |
|---|---|---|---|---|---|---|---|
| $\alpha$ And. | −·06 | −·05 | −·01 | $\delta$ Leo. | +·33 | +·31 | +·02 |
| $\gamma$ Peg. | −·29 | −·25 | −·04 | $\alpha$ CVn. (br.) | −·01 | +·05 | −·06 |
| $\delta$ Cas. | +·29 | +·36 | −·07 | $\eta$ UMa. | −·19 | −·16 | −·03 |
| $\beta$ Ari. | +·34 | +·36 | −·02 | $\gamma$ UMi. | +·23 | +·17 | +·06 |
| $\beta$ Tri. | +·30 | +·33 | −·03 | $\alpha$ CrB. | +·05 | −·02 | +·07 |
| $\delta$ Per. | −·07 | +·06 | −·13 | $\alpha$ Oph. | +·36 | +·30 | +·06 |
| $\eta$ Tau. | +·01 | +·03 | −·02 | $\alpha$ Lyr. | +·05 | +·05 | ·00 |
| $\beta$ Tau. | −·11 | −·06 | −·05 | $\zeta$ Aql. | +·07 | +·13 | −·06 |
| $\beta$ Aur. | +·11 | +·16 | −·05 | $\alpha$ Aql. | +·45 | +·40 | +·05 |
| $\alpha$ Gem. (mean) | +·15 | +·16 | −·01 | $\alpha$ Cyg. | +·32 | +·19 | +·13 |
| $\alpha$ Leo. | −·04 | +·01 | −·05 | $\alpha$ Cep. | +·50 | +·43 | +·07 |
| $\beta$ UMa. | +·03 | +·09 | −·06 | $\alpha$ Peg. | −·02 | −·08 | +·06 |
| | | | | Means | +·12 | +·12 | |

From this comparison we find that the probable error of the difference Greenwich − Hertzsprung is ± ·044. The mean weight given by Hertzsprung for these 24 stars is 249, corresponding to a probable error of ± ·054. It seems likely that Hertzsprung's probable errors have been overestimated, at any rate for the brighter stars.

To derive colour temperatures from the gradients the colour temperature corresponding to zero gradient must be known. The only extensive series of comparisons with a terrestrial source are those of Wilsing, on whose results Hertzsprung's data were calibrated. They give $\frac{c_2}{T} = 1{\cdot}48$ for G = 0 (see above), for which the corresponding T = 9670°. This value is generally believed to be too low, and we have followed Russell, Dugan, and Stewart in adopting

$$T = 11{,}000^\circ \text{ for mean A0},$$

*i.e.* for zero gradient. This adopted value has the merit of bringing the derived temperatures of the dwarf G0 stars (see Table IV. below) into sensible agreement with the generally accepted temperature of the Sun.

Using T = 11,000° for G = 0 the constant $k$ is − 1·40 and

$$G = -1{\cdot}40 + \frac{c_2}{T}\Big/\left(1 - e^{\frac{-2c_2}{T}}\right).$$

The colour temperatures of the standard stars as derived from the gradients by this equation are as follows :—

TABLE III.

*Temperatures of Standard Stars.*

| Star's Name. | $\frac{c_2}{T}$. | T. | Star's Name. | $\frac{c_2}{T}$. | T. |
|---|---|---|---|---|---|
| $\alpha$ And. | 1·23 | 11,600° | $\delta$ Leo. | 1·67 | 8,600° |
| $\gamma$ Peg. | 0·94 | 15,200 | $\alpha$ CVn. (br.) | 1·29 | 11,100 |
| $\delta$ Cas. | 1·62 | 8,800 | $\eta$ UMa. | 1·06 | 13,500 |
| $\beta$ Ari. | 1·68 | 8,500 | $\gamma$ UMi. | 1·56 | 9,200 |
| $\beta$ Tri. | 1·63 | 8,800 | $\alpha$ CrB. | 1·36 | 10,500 |
| $\delta$ Per. | 1·21 | 11,800 | $\alpha$ Oph. | 1·70 | 8,400 |
| $\eta$ Tau. | 1·31 | 10,900 | $\alpha$ Lyr. | 1·36 | 10,500 |
| $\beta$ Tau. | 1·16 | 12,300 | $\zeta$ Aql. | 1·38 | 10,400 |
| $\beta$ Aur. | 1·42 | 10,100 | $\alpha$ Aql. | 1·80 | 8,000 |
| $\alpha$ Gem. (mean) | 1·47 | 9,700 | $\alpha$ Cyg. | 1·66 | 8,600 |
| $\alpha$ Leo. | 1·25 | 11,500 | $\alpha$ Cep. | 1·86 | 7,700 |
| $\beta$ UMa. | 1·33 | 10,800 | $\alpha$ Peg. | 1·28 | 11,200 |

In addition to the standard stars, other stars have been under observation, and the results for 19 of these, each of which has been compared with at least two standards, are given in Table IV. In the case of the Go stars the gradients have been derived from measures covering the range $\lambda$ 4500 to $\lambda$ 6500.

TABLE IV.

*Colour Temperatures of Stars.*

| Star's Name. | R.A. 1925·0 | Dec. | Vis. Mag. | Spectral Type. (Harvard.) | No. of Obs. | Gradient. | $\frac{c_2}{T}$. | T. |
|---|---|---|---|---|---|---|---|---|
| | h m | ° ′ | m | | | | | |
| $\gamma$ Cas. | 0 52·2 | +60 19 | 2·3 | B0*p* | 17 | + ·12 | 1·43 | 10,000° |
| $\varepsilon$ Cas. | 1 49·0 | +63 18 | 3·4 | B3 | 6 | − ·17 | 1·09 | 13,100 |
| $\beta$ Per. | 3 3·3 | +40 40 | 2·3–3·5 | B8 | 12 | − ·01 | 1·29 | 11,100 |
| $\zeta$ Per. | 3 49·4 | +31 40 | 2·9 | B1 | 9 | + ·39 | 1·73 | 8,300 |
| $\varepsilon$ Per. | 3 52·8 | +39 48 | 3·0 | B1 | 10 | − ·19 | 1·06 | 13,500 |
| $\beta$ Leo. | 11 45·2 | +14 59 | 2·2 | A2 | 9 | + ·21 | 1·53 | 9,400 |
| $\gamma$ UMa. | 11 49·9 | +54 7 | 2·5 | A0 | 11 | ·00 | 1·30 | 11,000 |
| $\varepsilon$ UMa. | 12 50·7 | +56 22 | 1·7 | A0*p* | 13 | ·00 | 1·30 | 11,000 |
| $\beta$ Com. | 13 8·4 | +28 15 | 4·3 | G0 | 8 | +1·12 | 2·50 | 5,700 |
| $\zeta^1$ UMa. | 13 20·9 | +55 19 | 2·4 | A2*p* | 9 | + ·16 | 1·48 | 9,700 |

TABLE IV.—*continued.*

| Star's Name. | R.A. 1925·0. | Dec. 1925·0. | Vis. Mag. | Spectral Type (Harvard.) | No. of Obs. | Gradient. | $\frac{c_2}{T}$ | T. |
|---|---|---|---|---|---|---|---|---|
| | h m | ° ′ | m | | | | | |
| $\eta$ Boo. | 13 51·1 | +18 46 | 2·8 | Go | 9 | +1·08 | 2·46 | 5,800° |
| $\zeta$ Her. | 16 38·5 | +31 44 | 3·0 | Go | 7 | +1·28 | 2·67 | 5,400 |
| $\delta$ Cyg. | 19 42·6 | +44 57 | 3·0 | Ao | 11 | − ·02 | 1·28 | 11,200 |
| $\beta$ Del. | 20 34·0 | +14 20 | 3·7 | F5 | 6 | + ·96 | 2·33 | 6,100 |
| Pi. 332 | 20 43·5 | +57 19 | 4·6 | Go | 8 | +1·00 | 2·37 | 6,000 |
| $\beta$ Cep. | 21 27·7 | +70 14 | 3·3 | B1 | 13 | − ·28 | 0·95 | 15,100 |
| 9 Cep. | 21 35·9 | +61 45 | 4·9 | B2*p* | 7 | + ·75 | 2·11 | 6,800 |
| $\zeta$ Peg. | 22 37·7 | +10 26 | 3·6 | B8 | 8 | − ·09 | 1·19 | 12,000 |
| $v$ Peg. | 23 21·6 | +22 59 | 4·6 | Go | 6 | +1·17 | 2·55 | 5,600 |

It will be observed that among the stars of spectral types Bo to B2 the dispersion in gradient and temperature is very large. One of the stars, $\gamma$ Cassiopeiæ, has an abnormal spectrum characterised by strong emission lines of unusual complexity and breadth, and it would not be surprising for its colour temperature to be abnormal.

It has been pointed out in a previous note * that early-type stars with strong $Ca^+$ lines tend to be of comparatively low colour temperature, and two such stars, $\zeta$ Persei and 9 Cephei, appear in the above list. Two similar stars, $\chi_2$ Orionis and $\alpha$ Camelopardali, mentioned in the note referred to, have not been sufficiently observed, but from the available material their gradients are respectively + 0·80 from two observations and + 0·14 from four observations. The corresponding temperatures are 6600° K and 9800° K, and the spectral types are B2*p* and Bo.

The dispersion in colour temperature of these early-type stars has also been observed and discussed by B. P. Gerasimovič,† and the low colour temperature of $\zeta$ Persei has been the subject of a paper by H. Kienle.‡ Attention was directed to the existence of a group of yellow stars of early type by E. Hertzsprung in 1923.§

The degree of accordance shown by the individual observations in relation to the probable error of a single comparison is generally satisfactory. In the case of $\alpha$ Cygni, however, out of 39 residuals 8 exceed three times the probable error, and of these 3 exceed five times the probable error. As the plates are not of poor quality, it is hardly possible to ascribe the large residuals to this cause. Their occurrence may be accidental, but, on the other hand, gives rise to the suspicion that $\alpha$ Cygni may be variable in colour, the range of the variation being possibly 0·20 in gradient, which is equivalent to $0^m{\cdot}10$ in colour index.

* *M.N.*, **89**, 125, 1928.
† *Harvard Bulletin*, 864, and H.C. 339.
‡ *M.N.*, **88**, 700, 1928.
§ *B.A.N.*, **1**, 204, 217.

The existence of a system of standard stars with well-determined gradients will be of great assistance in the spectro-photometric determination of absolute colour temperatures of stars.

The temperature corresponding to zero gradient must be determined by comparison between stars and a terrestrial source, accompanied by observations of the coefficient of atmospheric reddening. The latter is a variable quantity, and determinations of its value should be made at the time of comparison and not by observations of the same star at different altitudes. The use of standard stars at high and low altitudes makes this possible.

It is hoped to undertake absolute determinations of stellar temperatures in the near future.

*Royal Observatory, Greenwich :*
*1929 November 6.*

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## *The Moving Foreground of the Large Magellanic Cloud.*

By W. M. Smart, M.A., D.Sc.

(*Communicated by the Director, Cambridge Observatory.*)

§ 1. *Summary.*—Recently Dr. W. J. Luyten of Harvard Observatory has published the proper motions of 500 stars, as faint as magnitude 15·5 (photographic), in the region of the Large Magellanic Cloud. The great majority of these proper motions are over 0″·03 per annum. In a later paper, Luyten has discussed the proper motions statistically, and the following general results emerge from his investigation : (*a*) the proper motions show very clearly the effects of streaming in the expected directions, (*b*) the proportion of Drift II. stars to Drift I. stars is apparently abnormally low.

In the present paper, the proper motions, neglecting the comparatively few under 0″·03 per annum, are submitted to a more detailed analysis, and the statistical effect of restricting the annual proper motions to those of 0″·03 and over is removed. It is then found that the proportion of Drift I. stars to Drift II. stars is 1·9 : 1, which is not very far removed from the usual proportion found in other parts of the sky.

The velocities of the true drifts (in which all values of the proper motions, great and small, are supposed to be included) are found to be entirely normal.

The last part of the paper deals with the distribution of the proper motions according to their magnitude, and the observational evidence is in accordance with the assumed formula for the distribution of stars in space, which I have considered in previous investigations.

§ 2. The plates on which Luyten's proper motions * are based were

* *Harvard Circular*, No. 326, 1928.

taken with the Bruce telescope. The interval is 30 years. Measures were made on at least two pairs of plates with blink microscopes, and four measurers were engaged in this part of the work. Only those proper motions that appeared to be established with most certainty have been published. In all, there are 500 stars in Luyten's list, and the great majority of the proper motions are over 0″·03 per annum.

Luyten remarks: * "As the relatively stationary background formed by the stars in the cloud provides an exceptionally good zero point from which to measure proper motions, it may be expected that the survey with the blink microscope is more complete than usual and thus better adapted to statistical discussion." The average mean accidental error of the annual proper motions derived from the comparison of the measures made on two pairs of plates is 0″·013.

Luyten first investigates possible systematic errors in the published proper motions (*Circular* 326). As the two star-drifts are very nearly in position angles 0° and 180°, the components of proper motion in right ascension are peculiarly suitable for deriving any systematic error depending on the positions of the stellar images on the plates and on differential coma. The proper motions, grouped according to quadrants, show such little systematic effect in right ascension that this possible source of error can be neglected. The components in declination, of course, cannot be treated in this way. The right ascension components also afford information as to magnitude error—the error in the proper motions depending on apparent magnitude. Luyten gives a table of corrections for the various magnitude groups. The corresponding corrections, due to magnitude error, in declination are derived from an inter-comparison with some results by Innes. The mean magnitude of the comparison stars is about 14·3; many of these are believed to belong to the cloud, and the correction to be applied to the relative proper motions to convert them into absolute proper motions is dependent on the parallactic motion of the stars not associated with the cloud and on the motion of the cloud itself. The correction is extremely small and is combined by Luyten with the magnitude errors already mentioned. I have applied this total correction to the 500 annual proper motions in *Circular* 326; of this number, 421 have a minimum value of 0″·03, and it is to these restricted motions that the investigation in this paper properly applies.

§ 3. The distribution of the 421 restricted proper motions according to position angle is shown in Table I. and illustrated by the full-line curve in fig. 1. The counts in the table refer to 30° sectors, of which the position angle indicated is the median.

* *Harvard Circular*, No. 328, p. 1, 1928.

TABLE I.

*Counts of Stars moving in 30° Sectors of Position Angle.*

| Position Angle. | Number of Stars. | Position Angle. | Number of Stars. | Position Angle. | Number of Stars. |
|---|---|---|---|---|---|
| 10° | 134½ | 130° | 17½ | 250° | 5½ |
| 20 | 111 | 140 | 17½ | 260 | 4 |
| 30 | 93½ | 150 | 25 | 270 | 4½ |
| 40 | 64 | 160 | 31 | 280 | 6 |
| 50 | 42½ | 170 | 33½ | 290 | 9½ |
| 60 | 25½ | 180 | 26 | 300 | 15½ |
| 70 | 18 | 190 | 17 | 310 | 23 |
| 80 | 12½ | 200 | 13½ | 320 | 38 |
| 90 | 9½ | 210 | 12 | 330 | 57½ |
| 100 | 10½ | 220 | 10½ | 340 | 83½ |
| 110 | 12 | 230 | 11 | 350 | 116 |
| 120 | 15½ | 240 | 6 | 360 | 130½ |

The frequency curve based on Table I. and shown in fig. 1 has been analysed into two drifts. The results are in Table II.

TABLE II.

*Analysis of the Frequency Curve.*

| | Drift I. | Drift II. |
|---|---|---|
| Drift-velocity, $hV_1$ | 1·5 | 0·9 |
| Position angle, $\theta$ | 5° | 170° |
| Number of stars, R($\mu$) | 320 | 101 |

The black dots in fig. 1 define the theoretical curve based on the drift-constants of Table II. It will be seen that the agreement between theory and observation is very satisfactory. Taking the co-ordinates * of the apices of Drift I. and Drift II. to be (90°, − 10°) and (285°, − 64°) respectively, I have computed the corresponding position angles of the two drifts (the co-ordinates of the Large Magellanic Cloud are 85°, − 69°) and the results are: 6° for Drift I., agreeing very closely with the position angle 5° derived from the frequency curve (line 2 of Table II.) and 192° for Drift II. The discrepancy between the calculated and observed values (192° and 170°) for the position angle of Drift II. is just a little higher than that usually encountered in this kind of investigation.

The most noticeable feature of the results given in Table II. is the apparent exceptionally high ratio (3·2 : 1) of the number of Drift I. stars to the number of Drift II. stars to which Luyten drew attention in general terms. Also the drift-velocities are apparently very much

* These are weighted means of the values from several investigations referred to in *M.N.*, **89**, 152 (Table V.), 1928.

higher than would be expected for this region. Actually, the angular distances of the centre of the region from the drift-apices are 59° and 134° respectively, and assuming the usual values of the *space-velocities* of the two drifts to be 1·5 and 0·9, the values we should at first sight expect from the analysis of the frequency curve are respectively 1·5 sin 59° and 0·9 sin 134° or, in round figures, 1·3 and 0·6. These last values are much smaller than those obtained from the analysis of

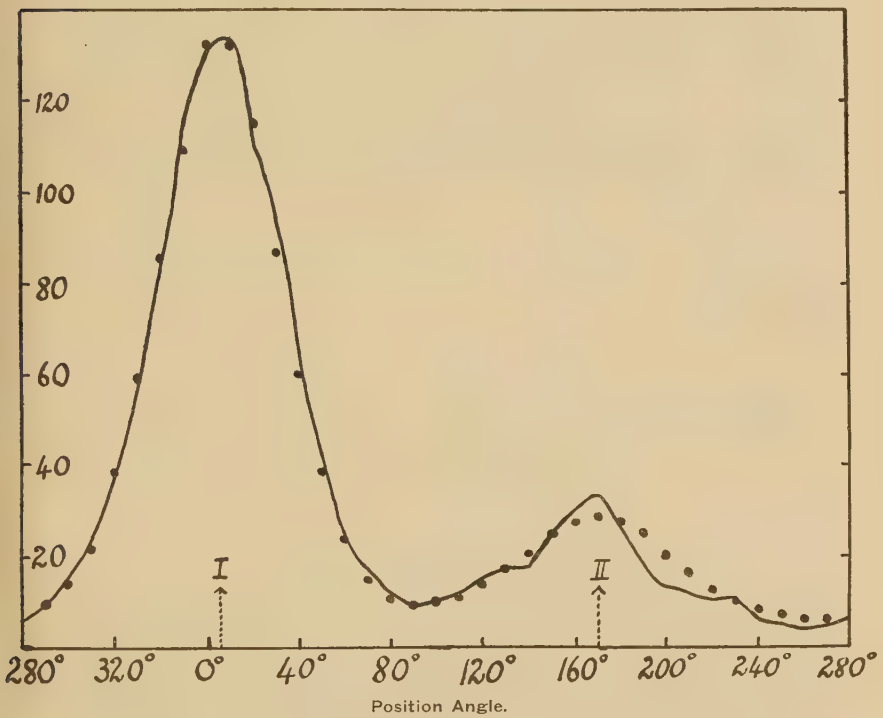


Fig. 1.—The full-line curve gives the *observed* distribution of proper motions in position angle; the series of dots indicates the *theoretical* distribution.

the frequency curve (Table II., line 1). We now explain these apparent anomalies, considering first the drift-velocities.

§ 4. It is to be remembered that the distribution of proper motions shown in fig. 1 is not the distribution given by two *true* drift-curves. A true drift-curve takes account of *all* the proper motions, great and small. In this investigation all the annual proper motions less than 0″·03 have been omitted, thus leaving what I have called a "distribution of restricted proper motions." In a previous paper * I have shown the relation between a true drift-curve and the pseudo-drift curve derived by restricting the proper motions to those exceeding a definite

* *M.N.*, **89**, 93, 1928.

value, on the assumption that the volume density $D(r)$ of the stars at a distance $r$ is given by

$$D(r) = \frac{Ae^{-h^2k^2r^2}}{r} \quad . \quad . \quad . \quad . \quad (1)$$

where $k$ is a parameter depending on the galactic latitude of the region concerned. The distribution in fig. 1 is thus a distribution given by two pseudo-drifts. The general effect of ignoring the small proper motions (up to $0''\cdot3$ per annum) is to make the frequency distribution of the restricted proper motions much more pointed than the corresponding true drift, and this leads at once, when an analysis is made, to apparently greater drift-velocities than would be derived if all the proper motions, great and small, were included. Thus the drift-velocities $h\mathrm{V}_1$ given in Table II. are greater than the velocities in the true drifts; the velocities $h\mathrm{V}_1$ are, in fact, the velocities given by the two pseudo-drifts consisting of stars with annual proper motions of $0''\cdot03$ and over.

Let $h\mathrm{V}$ denote the velocity of the true drift. Then I have shown * that the relation between the true-drift curve and the corresponding pseudo-drift curve is given by

$$\frac{e^{-h^2\mathrm{V}^2}\{f(\tau) - \eta^2 f(\eta\tau)\}}{1 - \eta^2 e^{-h^2\mathrm{V}^2(1-\eta^2)}} = e^{-h^2\mathrm{V}_1^2} f(\tau_1) \quad . \quad . \quad (2)$$

in which

$$\tau = h\mathrm{V}\cos(\theta - \theta_0)\,;\ \tau_1 = h\mathrm{V}_1\cos(\theta - \theta_0) \quad . \quad (3)$$

$$\eta = \frac{\mu}{(\mu^2 + k^2)^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and

$$f(\tau) = \frac{2}{\sqrt{\pi}}\left\{\tfrac{1}{2} + \tau e^{\tau^2}\int_{-\tau}^{\infty} e^{-x^2}dx\right\} \quad . \quad . \quad (5)$$

In (3) $\theta_0$ denotes the position angle in which the drift concerned is pointing.

The numerical values of $h\mathrm{V}_1$ corresponding to different values of $h\mathrm{V}$ and $k$ have been computed † when the limiting value of the annual proper motion is $0''\cdot02$. To make the results applicable to the present investigation where the lower limit of the annual proper motions is $0''\cdot03$, we can choose a new value of the parameter $k$—call it K—which will make the value of $\eta$ occurring in the principal formula (2) the same in the two cases. From (4) we then have

$$\eta^2 = \left\{1 + \left(\frac{k}{3\cdot0}\right)^2\right\}^{-1} = \left\{1 + \left(\frac{\mathrm{K}}{2\cdot0}\right)^2\right\}^{-1}$$

from which $\mathrm{K} = \frac{2}{3}k$.

The value of $k$ being supposed known, we can then make use of the table referred to by picking out the appropriate entry corresponding to the value of K.

* *M.N.*, **89**, 95, 1928. † *Loc. cit.*, p. 98 (Table II.).

The centre of the region of the sky concerned is in galactic latitude $-33°$, for which the value of the parameter $k$ is believed to be in the neighbourhood of 1″. Thus we can take $K = \frac{2}{3}$. Now the observed value of the pseudo-drift velocity $h\mathrm{V}_1$ for Drift I. has been found from the frequency curve (fig. 1) to be 1·5. Hence, interpolating in Table II. (*M.N.*, **89**, 98, 1928) with $K = \frac{2}{3}$, we infer that the true drift-velocity $h\mathrm{V}$ is 1·28. In the same way we find that for Drift II. the true drift-velocity is 0·68. It can be readily inferred that in the present instance a fairly large error in the assumed value of the parameter $k$ has a comparatively insignificant effect on the deduced true drift-velocities.

If $h\mathrm{W}$ denotes the true drift-velocity in space and $\lambda$ the angular distance of the region from the apex of a drift, we have

$$h\mathrm{W} = h\mathrm{V} \operatorname{cosec} \lambda \quad . \quad . \quad . \quad . \quad (6)$$

from which the values of the space-velocities of the two *true* drifts can be easily calculated. We summarise the results as follows :—

TABLE III.

| | Drift I. | Drift II. |
|---|---|---|
| Pseudo drift-velocity, $h\mathrm{V}_1$ (observed) | 1·5 | 0·9 |
| True drift-velocity, $h\mathrm{V}$ | 1·28 | 0·68 |
| $\lambda$ | 59° | 134° |
| Space-velocity of true drift | 1·49 | 0·94 |

In the last line of Table III. the values of the space-velocities of the true drifts are practically identical with the corresponding results of investigations in which true drifts alone are concerned—that is, where all proper motions, great and small, are used in the analysis.

In two previous papers * I have associated the theoretical forms of the restricted drift-curves with the observed distributions and derived the velocities of the true drifts. In these instances the mathematical theory seemed to be in accordance with the observations, for the final results were in close agreement with the true drift-velocities obtained in several other investigations in which no restriction is imposed on the values of the proper motions. As we have seen—as summarised in the last line of Table III.—the present investigation provides additional confirmation.

§ 5. As the probable errors of Luyten's proper motions are rather large, it was thought advisable to repeat the work already described, considering only annual proper motions of 0″·04 and over. The effect of accidental error in the distribution of proper motions must then be very much reduced. Three hundred and thirty-four proper motions are available. The analysis of the frequency curve resulted as follows :—

TABLE IV.

*Drift-Constants for Annual Proper Motions of 0″·04 and over.*

| | Drift I. | Drift II. |
|---|---|---|
| Pseudo-drift-velocity, $h\mathrm{V}_1$ | 1·5 | 0·9 |
| Position angle, $\theta$ | 5° | 170° |
| Number of stars, $\mathrm{R}(\mu)$ | 256 | 78 |

* *M.N.*, **89**, 93 ; **89**, 147, 1928.

The ratio of Drift I. stars to Drift II. stars is thus 3·3 : 1. When the space-velocities of the true drifts are calculated in the manner described for the proper motions of 0″·03 and over, there is no substantial departure from the values given in Table III.

§ 6. We now examine the second apparent anomaly in the distribution of Luyten's proper motions, and, as we shall see, it is mainly a consequence of the omission of the small proper motions. Let $R_1(\mu)$, $R_2(\mu)$ denote the observed numbers of stars forming the two pseudo-drifts I. and II. Then we have seen from our analysis of the frequency curve that

$$R_1(\mu) : R_2(\mu) = 3{\cdot}2 : 1$$

when the lower limiting value of $\mu$ is taken to be 0″·03 per annum, and just a trifle higher when the limiting value is taken to be 0″·04.

This ratio is apparently much too high, for in all previous investigations the numerical preponderance of Drift I. stars over Drift II. stars is very much less marked. If $N_1$ denotes the number of stars forming the true Drift I. (without any restriction as to the magnitude of the proper motions), I have shown * that $R_1(\mu)$ is related to $N_1$ by the formula

$$R_1(\mu) = N_1\{1 - \eta^2 e^{-h^2V^2(1-\eta^2)}\} \quad . \quad . \quad (7)$$

in which $h$V is the true drift-velocity and $\eta$ is given by (4), $\mu$ being the limiting value of the proper motions concerned. There is a similar relation between $R_2(\mu)$ and $N_2$, which refer to Drift II.

Taking Drift I. first, we have $h$V = 1·28 ; also assuming that $k = 1''$, we have $\eta^2 = 0{\cdot}9$ when the limiting value $\mu$ is 0″·03. We obtain from (7)

$$R_1(\mu) = 0{\cdot}240\, N_1 \quad . \quad . \quad . \quad . \quad (8)$$

Taking Drift II. next and assuming the same value for $k$, we obtain in a similar manner

$$R_2(\mu) = 0{\cdot}143\, N_2 \quad . \quad . \quad . \quad . \quad (9)$$

But we have found that

$$R_1(\mu) : R_2(\mu) = 3{\cdot}2 : 1.$$

Hence

$$N_1 : N_2 = 1{\cdot}9 : 1 \quad . \quad . \quad . \quad . \quad (10)$$

When the calculations are repeated with $\mu$ = 0″·04 as the limiting value of the annual proper motions, we arrive at a ratio almost identical with that given by (10).

The ratio which we have just derived for the numbers of stars in the *true* drifts is thus not so very discordant ; for the Boss stars, Eddington † found the ratio to be 1·5 : 1, this being the average for the whole sky ; for the Cambridge ‡ photographic proper motions of stars as faint as magnitude 12·0 the ratio was found to be 1·1 : 1, which was practically identical with the ratio given by the Groningen § proper motions.

It is to be noted that in the preceding calculations we have taken

* *M.N.*, **88**, 580, 1928.
† *M.N.*, **71**, 38, 1910.
‡ Smart, *M.N.*, **87**, 122, 1926.
§ Smart, *M.N.*, **88**, 144, 1927.

the same value of the parameter $k$ for the two drifts. This amounts to the assumption that the mean distances of the stars in the two drifts are the same. It might be expected from previous investigations that for very faint stars the ratio $N_1 : N_2$ would be more nearly equal to unity than that we have just found. If this is so, it follows that the mean distances of the stars in the two drifts cannot be the same, and that, consequently, the value of the parameter $k$ for Drift I. must be different from that for Drift II. I have made some other calculations for the ratio $N_1 : N_2$, which are shown in Table V.; the value of $k$ for Drift I. has been taken throughout to be 1″·0 and three values of $k$ for Drift II. are considered.

TABLE V.

*Values of the Ratio* $N_1 : N_2$.

| $k$. | | |
|---|---|---|
| Drift I. | Drift II. | $[N_1 : N_2]$. |
| ″ | ″ | |
| 1·0 | 1·0 | **1·9** |
| 1·0 | 0·9 | **1·6** |
| 1·0 | 0·8 | **1·2** |

A rough extrapolation from this table shows that the numerical strength of the two drifts will be equal when the value of $k$ for Drift II. is taken to be 0·75. As the mean distance * of a drift is inversely proportional to the value of $k$ for that drift, the stars in the two true drifts will be numerically equal provided it is assumed that the mean distance of Drift II. stars is about 30 per cent. greater than the mean distance of the Drift I. stars.

We summarise the conclusions of this section as follows: Apparently the distribution of stars between the two drifts is anomalous; this has been seen to be due to the restriction of the annual proper motions to those with values of 0″·03 or over. The mathematical theory is successful in showing that the proportion of Drift I. stars to Drift II. stars is practically normal when the limitation imposed on the values of the proper motions is withdrawn. Instead of being anomalous, the proper motions in the region of the Large Magellanic Cloud conform closely to the general behaviour observed in other parts of the sky.

§ 7. The absence of the small proper motions prevents us from adequately testing formula (1) for the distribution of the stars in space. It is reasonable to suppose that the number of stars with annual proper motions over 0″·05 is tolerably complete; these can then furnish a comparison with the computed numbers derived from the mathematical theory based on (1). Unfortunately, as we shall see, the large proper motions by themselves are unable to do more than indicate the general order of magnitude of the parameter $k$ in (1). If it were certain that the number of stars with annual proper motions of 0″·03 and over was complete the test could be made a more searching one; but it is fairly

* *M.N.*, **89**, 104, 1928.

obvious that this condition for Luyten's proper motions must be far from being realised.

I should like to take this opportunity of urging the importance of obtaining, with as high accuracy as possible, the proper motions of *all* the stars photographed in any region. In the search for the true law of stellar distribution, the stars with small proper motions—that is, the very distant stars—clearly play a very important part.

Let $hV_1$ and $hV_2$ now denote the velocities in the true drifts; $N_1$ and $N_2$ the numbers of stars in the true drifts; $R_1(\mu)$ and $R_2(\mu)$ the total number of stars with proper motions of $\mu$ and over. Then, if $\rho(\mu)$ denotes the sum of $R_1(\mu)$ and $R_2(\mu)$, we have by (7)

$$\rho(\mu) = N_1\{1 - \eta_1^2 e^{-h^2V_1^2(1-\eta_1^2)}\} + N_2\{1 - \eta_2^2 e^{-h^2V_2^2(1-\eta_2^2)}\} \quad . \quad (11)$$

where

$$\eta_1^2 = \frac{\mu^2}{\mu^2 + k_1^2} \quad \text{and} \quad \eta_2^2 = \frac{\mu^2}{\mu^2 + k_2^2} \quad . \quad . \quad (12)$$

in which $k_1$ and $k_2$ are the values of the parameter $k$ for the two drifts.

In § 6 we found that if $k_1 = k_2 = 1''{\cdot}0$, the observed proper motions led to the ratio $N_1 : N_2 = 1{\cdot}9 : 1$. We can then test the observed values of $\rho(\mu)$ against the theoretical values obtained from (11). Denoting the coefficients of $N_1$ and $N_2$ in (11) by $A_1$ and $A_2$ we have

$$\rho(\mu) = N_2\left\{\frac{N_1}{N_2}A_1 + A_2\right\} \quad . \quad . \quad . \quad (13)$$

For $\mu = 0''{\cdot}05$, we find that $A_1 = 0{\cdot}0982$, $A_2 = 0{\cdot}0557$, and inserting the value of $N_1/N_2$ we obtain

$$\rho(\mu) = 0{\cdot}243\ N_2.$$

The observed number of stars with annual proper motions of $0''{\cdot}05$ and over is 240. To obtain a convenient comparison between the theoretical values of $\rho(\mu)$ and the observed values we choose $N_2 = 988$. The following table gives the comparison of the observed values of $\rho(\mu)$ with the theoretical values computed for the three values of $k$: $0''{\cdot}0$, $1''{\cdot}0$, and $3''{\cdot}0$.

Table VI.

| $\mu$. | $\rho(\mu)$ (observed). | $\rho(\mu)$ (theoretical). | | |
|---|---|---|---|---|
| | | $k=0''{\cdot}0$. | $k=1''{\cdot}0$. | $k=3''{\cdot}0$. |
| $''$ | | | | |
| 0·05 | 240 | 240 | 240 | 240 |
| ·06 | 174 | 167 | 172 | 192 |
| ·08 | 93 | 94 | 97 | 127 |
| ·11 | 45 | 50 | 52 | 84 |
| ·14 | 26 | 30 | 32 | 49 |

If the ratio $k/\mu$ is small, it is easily shown that the formula (11) becomes simply

$$\rho(\mu) = \frac{B}{\mu^2} \quad . \quad . \quad . \quad . \quad (14)$$

If $k$ does not exceed 1″ for each drift, then for the range of $\mu$ considered (0″·05 to 0″·14) the curve given by (11) approximates very closely to that given by (14). This is shown by the entries in the third and fourth columns of Table VI. The observed values of $\rho(\mu)$ are in good agreement with the entries in the third or in the fourth column, and it is impossible to deduce a definite value of $k$ from the observations. This must be inevitably so when only large proper motions are used in the analysis. When we consider the entries in the last column, it is immediately apparent that although we cannot derive a value of the parameter $k$ we can nevertheless fix an approximate upper limit. This limit seems to be in the neighbourhood of 1″·0, for the divergence between the observed values of $\rho(\mu)$ and the theoretical values increases very markedly as $k$ increases from 1″·0 to 3″·0. If our proper motions had had a lower limiting value of 0″·01 per annum, the theoretical curve $y = \rho(\mu)$ would have been extremely sensitive to changes in $k$, especially in the range $0''{\cdot}01 < \mu < 0''{\cdot}05$, and we might have expected under these circumstances to have been able to fit one of the theoretical curves, based on (11), to the observed curve. It is just here that accurate small proper motions would have played a decisive rôle. With the present actual limitations of the observational material the evidence suggests that the upper limit to the value of $k$ is in the neighbourhood of that adopted.

§ 8. *Conclusion.*—The apparent anomalies in the observed distribution of Luyten's proper motions have been removed by considering the theoretical effect of excluding from the mathematical analysis all proper motions less than a definite amount. In this way it has been found that the true drift-velocities are entirely normal, and that the relative numerical strengths of the true drifts are very nearly so. Further, the observed distribution, according to magnitude, of the large proper motions (0″·05 and over) is at the worst not antagonistic to the assumed formula for the space distribution of the stars, and may even be claimed to offer substantial support to it.

*Observatory, Cambridge :*
1929 *October* 31.

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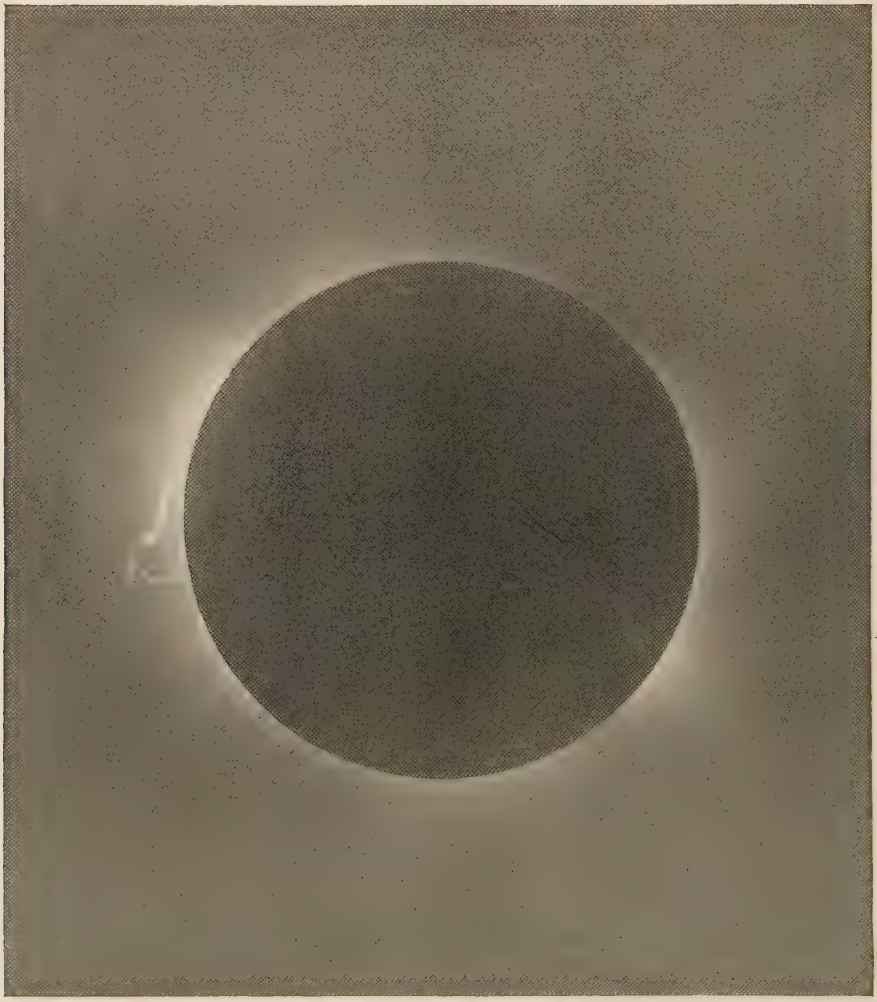
*Total Solar Eclipse of* 1929 *May* 9. *Reports of the British Expeditions.* (Plates 2–5.)

The eclipse of the Sun which took place on May 9 had the long duration of 5 minutes on the central line, and the track was such that fairly accessible stations could be selected from which the Sun would be seen high in the sky. In these respects the eclipse was more favourable than any to take place for a considerable number of years. It was early decided that a new attempt should be made to test the Einstein displacement, as the test could only be made during a long eclipse, and the Admiralty agreed to send Dr. Jackson of the Royal Observatory, Greenwich, for this purpose. The Solar Physics Observatory, Cambridge, undertook to send Dr. Carroll to make spectroscopic observations,

particularly a study of the corona with an interferometer and a comparison of the H and K lines of calcium with the infra-red group X. The conditions of the eclipse were reviewed by the Joint Permanent Eclipse Committee, and it was decided to approach the Government Grant Committee of the Royal Society with a view to obtaining funds to duplicate the observations. A grant was obtained, and it was decided to invite Professor F. J. M. Stratton of Cambridge and Mr. P. J. Melotte of Greenwich to take part in the expeditions. The programme of observations to be attempted was discussed with most of the intending observers from Europe at the Meeting of the International Astronomical Union at Leiden in 1928 July.

Col. J. Waley Cohen, who had assisted the eclipse expedition to Benkoelen in 1926, very kindly visited Malaya and Siam in 1928, travelled through the belt of totality investigating sites on behalf of the observers, and interested those in authority in the intended visits of the eclipse observers. From his report, in conjunction with meteorological records kindly secured by Mr. V. A. Löwinger, the Surveyor General to the Federated Malay States, it was decided to locate the observing parties at Alor Star in Kedah and Pattani in Southern Siam. As the eclipse took place near the time when the monsoon usually changes, it was hoped that this choice of stations on the two sides of the peninsula would ensure that one, at least, of the expeditions would meet with favourable conditions. The meteorological records showed that good conditions were more likely in the coastal areas than in the higher regions in the backbone of the peninsula. The stations selected were easily within a day's journey of each other, and the observing parties could keep in touch with one another. Dr. Jackson and Dr. Carroll were stationed at Alor Star and Professor Stratton and Mr. Melotte at Pattani. Dr. T. A. Royds, Director of Kodaikanal Observatory, was sent by the Indian Government to observe the eclipse, and joined the observers in Siam. It was known that a number of volunteer observers, including Dr. Aston at Alor Star and Col. Waley Cohen at Pattani, had offered to join the observing parties, and that a considerable amount of assistance in the manipulation of the instruments at the time of the eclipse might be obtained locally. Consequently, if the apparatus could be collected and adapted in England, a considerable programme could be undertaken.

The determination of the exact value of the Einstein displacement is a matter of great difficulty. There is no rich field of stars at any point on the ecliptic except at the point where the Sun is on May 29—a circumstance which made the eclipse of 1919 so favourable. On May 9 the field is a fair average one. The difficulty in applying the test is in disentangling the scale values from the true displacement. For this eclipse it was decided to adopt the plan of photographing a second field of stars on the eclipse plates during the eclipse itself, and a comparison field in the same right ascension as the Sun, but 9° further north, was selected. It was estimated that a ninefold increase in accuracy would be obtained in this way, as compared with exposures on the eclipse field only.




8s exposure, beginning 42s after beginning of totality.
Plate taken with 6-inch lens of 45 feet focal length at Alor Star.

$10^s$ exposure, beginning $20^{03}$ after beginning of totality.
Plate taken with 6-inch lens of 45 feet focal length at Alor Star.

By the kind permission of the Astronomer Royal and Professor Newall the work of preparing the instruments was undertaken at the workshops at Greenwich and Cambridge. The work at Greenwich was particularly heavy, as it involved the whole astrographic telescope which in itself weighed about 4 tons. The instruments were despatched from Greenwich on February 12 for shipment by s.s. *Laomedon* from Liverpool on February 16. Generous assistance was received from the Blue Funnel Company, who once again took out all the eclipse apparatus to Penang (or Singapore) and back free of all charge; and also from the Straits Steamship Company and the Siam Steamship Company, who carried the goods free from Penang to Alor Star and from Singapore to Pattani respectively and back again. Both parties of observers received many kindly services from officials of the Asiatic Petroleum Company, especially from Mr. Sturt at Penang; Mr. C. Wurtzburg at Singapore and Mr. Penrice at Penang of Messrs. Mansfield & Co. were very helpful in making shipping and transport arrangements.

### (*a*) Report by Dr. Jackson.

For the eclipse at Alor Star I had three instruments. (1) Photographic telescope with a two-lens objective of 7 inches aperture and 21 feet focus, fed by cœlostat, to test the Einstein theory; (2) Mr. Worthington's 6-inch Clark lens of 45 feet focus, also fed by cœlostat, for large-scale photographs of the corona; and (3) Dallmeyer Equatorial carrying three short-focus cameras for photography of the corona through colour screens.

Col. Cohen arrived in Malaya at the end of 1929 February and made preliminary arrangements before Mr. Melotte, Mrs. Jackson and I arrived at Alor Star on March 16. We proceeded to set up the Einstein telescope as quickly as possible in the hope of photographing the eclipse field at an altitude of about 30° to 35° immediately after sunset. This was desirable as the lens had been delivered so near the time of our departure from England that it had been impossible to test it adequately. However, the weather was so consistently bad immediately after sunset that it was impossible to do this. Indeed, in the six weeks before the eclipse it was difficult even to get the few photographs required for focus purposes. On the day of the eclipse the sky was covered by high cirrus cloud through which the Sun was dimly seen. The conditions were so poor that all the exposures were made from the computed times by chronometer and not from observation of the beginning of totality.

*The Einstein Camera.*—As it was desired to photograph two fields on the same plate, and as the telescope was fed by a cœlostat, a new arrangement had to be devised. We could move either the cœlostat or the telescope. A movement of the cœlostat could only give us a second field in the same declination as the eclipse field, and apart from the fact that no very suitable field could be found by moving in right ascension only it was considered inadvisable to interfere with the cœlostat during the eclipse itself. (The cœlostat, however, was provided with an hour circle which proved invaluable for picking up star fields

at night time.) It was therefore necessary to move the telescope. It would have been possible to arrange for any field to be photographed by a movement of the telescope in a suitable plane, but as a good field was found at the same right ascension, it was only necessary to move the telescope in an hour circle. One objection which can be raised against the use of a cœlostat is that a slightly different ellipse of the mirror is used in the two exposures. Care was taken to ensure that the telescope was so pointed that the ellipses used in the two exposures had the same centre. The following arrangement was used. The telescope was supported on two stands whose upper surfaces were in the plane of the desired hour circle, inclined at an angle of 6° 11′ to the horizon. These stands carried concentric circular rails of 21 feet and 2 feet radii respectively, and on them rested the wheels of the frames carrying the telescope. The stands were so arranged that the common centre of the two circular rails was in the centre of the cœlostat mirror. By means of a system of wires working on pulleys and coupled on a shaft with wheels in a ratio of 21 to 2, the telescope could be moved easily and smoothly along the rails on which it was supported. Stops were provided so that the telescope could be moved rapidly and with certainty from the one declination to the other. For the two fields photographed at the eclipse the displacement at the object-glass end was 4 inches and at the plate end 3 feet 4 inches. The longer rail was about 10 feet in length so that the stands could be accurately adjusted on the Sun several weeks before the eclipse. Heavy concrete blocks were tied to the stands to prevent any danger of movement when the telescope was moved. It might be mentioned here that a similar arrangement could be used to enable the same telescope to photograph the flash spectrum through an objective prism, and then—by removal of the prism and movement of the telescope—to secure direct photographs of the corona.

The programme was as follows, the times being counted from the beginning of totality:—

| | | s | s |
|---|---|---|---|
| 1st Plate. | Comparison Field . . | 5 to | 25 |
| | Eclipse Field . . | 35 ,, | 60 |
| 2nd Plate. | Eclipse Field . . | 75 ,, | 100 |
| | Comparison Field . . | 110 ,, | 135 |
| 3rd Plate. | Comparison Field . . | 150 ,, | 175 |
| | Eclipse Field . . | 185 ,, | 210 |
| 4th Plate. | Eclipse Field . . | 225 ,, | 250 |
| | Comparison Field . . | 260 ,, | 285 |

The programme was carried out as arranged, but all the plates are heavily fogged and no stars are to be seen on them. The great prominence on the eastern limb shows best on the plate developed for the shortest time. On the fully developed plates the scattered light prevents any real detail from being seen or the form of the corona from being made out.

The plates were 15 inches square and $\frac{3}{16}$th-inch thick and were backed. On account of the faintness of the stars a Monarch emulsion was used.

45-*foot Telescope.*—Mr. James Worthington, who had lent his 6-inch Clark lens of 45-foot focus for the eclipse of 1927, kindly extended the loan for this eclipse. On this occasion it was fed by a 16-inch cœlostat. The plates used were Ilford Special Rapid (backed), 12 by 15 inches, and the telescope was so mounted that the long side of the plates was vertical, agreeing roughly with the expected equatorial extension of the corona. The plates were exposed as follows, the times being counted from the commencement of totality:—

| | s | | s |
|---|---|---|---|
| 1. | 3 | to | 6. |
| 2. | 42 | ,, | 50. |
| 3. | 100 | ,, | 120. |
| 4. | 200 | ,, | 210. |
| 5. | 287 | ,, | 290. |

Actually the $20^s$ exposure does not show greater extensions of the corona than was expected from $3^s$ in a clear sky, and is about equal to that obtained at Giggleswick in 1927 with a similar exposure and the Sun at an altitude of only 11°. For the longest exposure there is a considerable amount of scattered light within the limb of the Moon. At the time the plates were exposed it was not expected that they would be anything like as good as they actually turned out to be, and just before totality the question of lengthening the exposures was considered. From these plates, in conjunction with those taken with the 21-foot, it appears that it is not advisable to prolong the exposure when photographing an eclipse through moderate cloud. The short exposures show the more brilliant features, while longer exposures do not bring up the fainter extensions, but merely cover the plate with scattered light from the brighter portions, and generally fog the plate.

The definition was excellent, as is shown by the fact that the same irregularities of the Moon's limb are shown on all the plates, independently of the brightness of the coronal background or the length of the exposure. There is a very large filamentous prominence on the eastern limb, and on the first plate, taken immediately after the Moon covered this limb, it is seen that there was slight continuous connection of the prominence with the Sun's limb. A considerable difference in the structure is shown on the first and last plates separated by 284 seconds. There is a fine coronal arch on the western limb. The plates accompanying this paper were made from these photographs.

*Dallmeyer Equatorial.*—At no recent eclipse has an attempt been made to photograph the greatest possible extent of the corona. In discussing the programme the idea occurred that the zodiacal light might be the outermost portion of the corona, and that it might be possible by the use of red sensitive plates and colour screens to photograph the corona during the eclipse and the zodiacal light soon after sunset. The regularity with which the sky clouded up before sunset showed the

latter proposal to be out of the question. Three cameras were mounted on one of the Dallmeyer equatorials which had originally been constructed for the Transit of Venus in 1874. The lenses used were :

1. Dallmeyer of 4 inches aperture and 32 inches focus.
2. Ross of 4 inches aperture and 16 inches focus.
3. Aldis of $3\frac{1}{2}$ inches aperture and 20 inches focus. (Lent by Dr. G. Merton.)

For the Dallmeyer lens, panchromatic plates (backed) were used, and for the others kryptocyanin, unbacked. For all exposures, except one, Wratten F Filters No. 29 were used, cutting out all light shorter than 6000 A. The other filter used was a Wratten No. 21, cutting out light of wave-lengths less than 5300 A., but as this was used in conjunction with a kryptocyanin plate it was effectively the same as the others. The filters were placed immediately in front of the plates, and for the Aldis lens the filters were actually in the plate holders. The effective wave-lengths might be taken as 6300 A. for the panchromatic plates and 7400 A. for the kryptocyanin plates.

In order to prevent shaking the telescopes in changing plates, the same exposures were given with the various cameras, viz. $180^s$ and $60^s$, beginning respectively $10^s$ and $215^s$ after the beginning of totality. A short description of the photographs may now be given.

The scattered light is much more troublesome on the panchromatic plates than on the others. It might at first sight be thought that this should be attributed to the shorter wave-length. It is, however, possible that the difference is, like the difference between the photographs taken with the 21-foot and 45-foot, due to difference in effective exposure. Although the ratio of aperture to focal length was less for the Dallmeyer lens than for the others, the speed of the panchromatic plate is much greater than that of the kryptocyanin. The panchromatic plates when examined with an eyepiece show the great prominence and some smaller ones, and a tendency for the coronal light to collect into rays. The total extension is just over a diameter from the limb, but the outer edges of the image are rather uniform in character, and probably a good deal of the light is " cloud-scattered " from the bright prominences and inner corona. Cloud formation is seen on the plates.

The photographs obtained from the two cameras using kryptocyanin plates show the form of the corona much more clearly. There are a number of streamers shown up to a distance of fully a radius from the limb and the corona is clearly of intermediate type.

For time purposes two chronometers were taken from England. The errors of these were kindly supplied by Capt. Nares of H.M. Survey Ship *Iroquois*, then stationed at Penang, some five days before the eclipse. Thereafter, Mr. Horn of the Perth Observatory, Australia, who had arrived with a wireless receiving set, got time from the Bordeaux and Malabar signals daily. As already stated, the exposures were made from the computed times, as no accurate commencement

Eastern limb of the Sun, exposure 3$^s$, beginning 3$^s$ after commencement of totality.

Western limb of the Sun, exposure $3^s$, ending $4^s$ before the end of totality.

of totality could be observed. The geographical co-ordinates of the camp were :

Longitude, 100° 22′·4 E. Latitude, 6° 8′·5 N.,

and the computed times of the beginning and end of totality :

$6^h\ 34^m\ 54^s$. $6^h\ 39^m\ 48^s$.

In a tropical country where the dew-point hovered round about 80° F. special difficulties in photography were encountered. A plentiful supply of ice was available and the photographic solutions were kept in an ice-chest. The plates were developed at 65° 70° in ordinary pyro-soda, and hardened for 10 to 15 minutes in a 5 per cent. bath of formalin. The washing had to be very thorough, as the least trace of hypo would prevent drying altogether. The plates were developed at night in the open, but they never dried till the following day, even when put in a cupboard with dishes of calcium chloride. The air was so moist at night that a fan was of little use in attempting to dry the plates.

It is a pleasure to record our thanks to H.H. The Regent of Kedah, the Kedah Government, and the British residents in Alor Star. The Government very generously voted a sum of money which covered all the expenses of the camp for both labour and material. A piece of public land outside the British Residency was placed at our disposal for the erection of our instruments. The British Adviser, the Hon. T. W. Clayton, literally placed his home at the disposal of the astronomers observing at Alor Star or passing through it. It was an enormous convenience in a tropical climate to live so near the observation camp and to have running water available for photographic purposes. The Public Works Department was constantly of great assistance to us in all the work of the camp and in the repairing of slight damage to instruments. To it, and especially to Mr. Jenkins and Mr. Davison, our thanks are due. The police did everything necessary for the protection of our instruments, and there was no trouble of any kind. During the eclipse itself I had the assistance of Mr. Allen and Mr. Scott in handling plates at the 21-foot telescope. The exposures with the 45-foot telescope were made by Mr. Bedlington, the surveyor to the Kedah Government, with two assistants to help in handling the plates. At the Dallmeyer equatorial Mrs. Jackson was assisted by Mr. and Mrs. Franklyn Robinson. The time was counted by Mr. Walker, assisted by Mr. Ellis. On account of the long duration of totality these gentlemen counted alternate hundreds by means of a rated metronome and checked the counting at intervals by means of a stop-watch started at the beginning of totality.

### (*b*) Report by Dr. Carroll.

It was decided that the Solar Physics Observatory, Cambridge, would undertake observations of the total eclipse of 1929 May 9, and that Dr. J. A. Carroll should have charge of the design, erection, and opera-

tion of the instruments for the following programme of observations to be carried out at Alor Star :—

*Spectrophotometric Observations on the Chromosphere.*

1. A three-prism spectrograph was arranged to obtain a record of the flash spectrum on a moving film in the visible and near ultra-violet regions. In this arrangement a radial slit and small solar image were used, and the camera contained a plane parallel glass plate to give two images of the spectrum so that correct exposures for both the strong and weak lines could be given (*M.N.R.A.S.*, **87**, 679).

   This instrument was operated at the eclipse by Dr. J. A. Carroll.

2. A grating spectrograph to record the infra-red chromospheric spectrum in the region of the ionised Calcium X lines ($\lambda\lambda$ 8498, 8542, 8662) on a moving plate, used also with a radial slit and so small a solar image that a second slit in the focal plane was not required.

   This instrument was operated at the eclipse by Mr. C. Jenkins, the Executive Engineer for N. Kedah.

   With these instruments the comparison of the H and K lines and the X lines of Calcium could be made.

*Interferometer Study of the Corona.*

3. An improved and enlarged objective interferometer similar to the one designed for the Norwegian expedition of 1927 June 29 was built at the Solar Physics Observatory. With this instrument velocities in the coronal gases could be studied by means of the monochromatic image in green light, and the separation of the interferometer plates was chosen also with the object of determining whether the green line was a close doublet, as suggested by Freeman.

   This instrument was operated at the eclipse by Dr. F. W. Aston.

*The Coronal Continuous Spectrum.*

4. A low dispersion, large aperture spectrograph working at $f/5$ was borrowed from Mr. Wilfred Hall and adapted for a photometric study of the coronal continuous spectrum over the whole visible and near infra-red regions.

   This instrument was operated at the eclipse by Mr. Horn and Mr. Bicknell.

Erection of the instruments was commenced on the site selected in Alor Star on March 28, and proceeded under rather trying conditions but without serious hindrance, thanks largely to the very generous and skilful help given by the Public Works Department of the Government

of Kedah and other public and private bodies, in particular Messrs. Huttenbach Lazarus, who installed the electrical supply, and the Penang Municipal Tramways.

Great difficulties were experienced owing to rain and especially to the excessively high humidity, whereby damage to the instruments arose and photographic work was particularly difficult. The plates for the infra-red spectrograph which had to be hypersensitised shortly before use could only be prepared with the aid of phosphorus pentoxide as a drying agent. Plates hypersensitised with borax were tried and proved satisfactory, but were not so sensitive as those treated with ammonia. The sliding parts of the moving-plate camera gave difficulty, and ultimately the camera was rebuilt on the spot and an endless travelling canvas band to which the plates could be attached was substituted for the sliding-plate holder. Mr. Davison of the Public Works Department was exceedingly helpful in this matter, and the revised camera proved entirely satisfactory.

Light clouds obscured the Sun at eclipse time and so no photographs susceptible of precise measurement were obtained with any of the instruments. Traces on the moving film camera were sufficient to demonstrate its suitability for the purpose, but it was not possible to make any use of them since no comparison spectra could be obtained on the day of the eclipse, and by the following day the dampness had damaged the film enough to ruin the regularity of any records there might have been. It is worth remarking that for work of this kind a particularly clear sky is absolutely essential, otherwise no reliable allowance can be made for the absorption of the earth's atmosphere.

The results obtained from the interferometer are regarded as highly profitable. The light was not strong enough to yield an image of measurable density, but there is an unmistakable impression on the plates. We are now in the position of being able to estimate accurately the exposures required for this experiment, and to feel justified in expenditure on larger and more elaborate apparatus. Previously it was entirely unknown whether it was possible to make such measurements at all.

### (*c*) Report of the Expedition to Pattani.

Col. Waley Cohen arrived at Pattani on March 21 and at once began work on the erection of pillars and shelters on the site close to the River Pattani, which had been allotted by the Siamese authorities. Dr. Royds arrived on March 30 and Messrs. Melotte and Stratton on March 31. Later the observing party was strengthened by the arrival of Professor E. Barnes and Professor W. F. Kibble from the Free Christian College, Madras, and Capt. E. G. Sinnott, Royal Signals, Lieut. W. M. Blagden, R.E., and Lieut. J. K. Haynes, R.F.A., from Singapore—a body of volunteers all of whom did excellent work in preparation for the eclipse and assisted later in the work of packing the instruments. Two or three days before the eclipse Professor Parames Waran of the Presidency College, Madras, and Dr. Ventakes Waran arrived and set up their

instruments (a 4-inch telescope and camera and a constant deviation spectrograph fed by a mirror) in the eclipse camp. The party was completed by the arrival on May 8 of Mr. C. Wingfield, British Minister at Bangkok, Mr. Coulson, District Officer at Khota Baru in Kelantan, and Mr. Abel Smith.

The instruments set up for the eclipse and the work proposed to be done with them were as follows :—

1. The Astrographic Refractor from Greenwich mounted equatorially, to be used for a redetermination of the amount of the displacement of the stars in the immediate neighbourhood of the eclipsed Sun. The stand used at Christmas Island in 1923 was employed, being suitable for the latitude of the observing station. This instrument was in charge of Mr. Melotte.

The scheme of observation to be carried out with the Astrographic Refractor was similar to that undertaken by Dr. Jackson at Alor Star, and is described in his report. The instrument was taken out complete and mounted equatorially though without the electric clock control. In view of the shortness of the exposures to be given (15 seconds) the electric control was not essential, as the governor of the clock is itself capable of very accurate adjustment. The photographic telescope is of 13 inches aperture and 135 inches focal length. For the eclipse plates the aperture was reduced to 7 inches in order to minimise the effect of sky fog. The telescope was housed in a shelter 30 by 15 feet, having a sliding roof covered with a tarpaulin 17 feet square. By giving a slope to the top rails, which were $11\frac{1}{2}$ feet above ground at the north end and $7\frac{1}{2}$ feet above ground at the south end, the sliding frame was easily drawn to one side clear of the instrument. This shelter was constructed at Greenwich of light wood, held together by bolts, and was erected in a day. The telescope was mounted, with the assistance of native labour, in seven days.

The plate centres of the two fields to be photographed were as follows :—

| | | |
|---|---|---|
| Eclipse Field . . . | R.A. $3^h\ 0^m\ 50^s$ | Dec. $+\ 17°\ 0'$. |
| Comparison Field . . | $3^h\ 0^m\ 50^s$ | $+\ 26°\ 7'$. |

The settings were carefully chosen in such manner that the stars in the immediate neighbourhood of the Sun were distributed as uniformly as possible with respect to the plate centre. The fact that the eclipsed Sun's image fell to one side was immaterial.

The setting for the comparison field was similarly chosen. In order to avoid loss of time in setting the telescope, a field having the same right ascension as the eclipse field was selected. These two fields gave a number of stars at about equal distances from the plate centre, and with stars of the comparison field falling near to stars in the eclipse field.

The programme of exposures was as follows :—

| | | | |
|---|---|---|---|
| Plate 1. | Eclipse Field . . | Exposure 10 seconds. | |
| | Comparison Field . . | ,, 10 | ,, |
| Plate 2. | Comparison Field . . | ,, 15 | ,, |
| | Eclipse Field . . | ,, 15 | ,, |

| | | | |
|---|---|---|---|
| Plate 3. | Eclipse Field . . | Exposure 15 seconds. | |
| | Comparison Field . . | ,, 15 | ,, |
| Plate 4. | Comparison Field . . | ,, 10 | ,, |
| | Eclipse Field . . | ,, 10 | ,, |
| Plate 5. | Eclipse Field . . | ,, 10 | ,, |
| | Comparison Field . . | ,, 10 | ,, |

Backed Ilford Special Rapid Plates were used. These were 8 inches square and coated on plate glass $\frac{3}{8}$-inch thick. During the rehearsals which took place before the eclipse the above programme was easily carried out within the limits of time available, including the accurate pointing of the telescope in declination by reading the circle at each change of setting. Thirty seconds before the time of second contact the telescope was pointed on the eclipse field. Although the sky was then completely overcast the diminishing crescent of the Sun could be dimly seen in the 2-inch finder attached to the instrument. Owing to the bad conditions no exposures were made during the first four minutes of totality. During the last minute two plates were exposed for 15 seconds each, on the eclipse field. The outline of the eclipsed Sun and the large prominence are faintly shown on these.

It is a pleasure to acknowledge the help received from Nai Chamlong Asavasena who assisted most efficiently throughout the eclipse, both in setting the telescope and changing plate holders. We are also glad to acknowledge the assistance afforded by Mr. T. Piper of the Pattani Tin Mines.

2. A Littrow spectrograph with a plane grating fed by a 16-inch cœlostat and a 15-inch mirror of 29-foot focal length. This instrument was designed by Mr. C. R. Davidson, built in the Royal Observatory workshop, and set up in the same way as at the Greenwich expedition to Giggleswick. The first order of the flash spectrum about 8800 A. was to be photographed through a red screen on half a plate dyed with neocyanin, while the second order of H and K was photographed through a green screen on the other half, for a comparison of the behaviour of $Ca^{+}$. If the observation was successfully made at the first flash, it was proposed to remove the screen, reset and refocus the grating, and photograph the second flash over the region $H\beta$ to $H\alpha$ on a panchromatic plate. Professor Stratton, assisted by Mr. Coulson, had the charge of this instrument. For this and the next instrument it was proposed to use the solar spectrum, secured just after the eclipse, for spectrophotometric comparison. The solar light was to be cut down in intensity by passage through ground quartz. The two spectra to be compared were to be taken on halves of the same plate and were to be developed together.

3. The Hills quartz spectrograph fed by a cœlostat with a 12-inch speculum flat and a 9-inch concave speculum mirror was in charge of Dr. Royds. The arrangement was much as at the 1926 eclipse, and the programme paid special attention to the Balmer series and to the distribution of light in the coronal spectrum, both the line and the continuous spectrum.

4. The 19-foot coronagraph fed by the 8-inch cœlostat and the 4-inch lens belonging to the Royal Irish Academy. As in the 1926 eclipse, the direct-vision prism of the Royal Observatory, Edinburgh, was to be placed in front of the lens for the first and second flash to give the spectrum in the region from H and K to $H\beta$. As the 8-inch cœlostat is constructed only for fairly high latitudes some alterations were necessary. It can be adapted easily to southern latitudes as the clock work can be easily levelled in a reversed position. It is impossible to put a wedge under the base and level the clock without reversing the motion of the drive. Consequently the cœlostat was erected in the position corresponding to a negative southern latitude, by tilting the base, so that what is usually the upper end of the polar axis pointed downwards to the south pole, and reversing the clock drive. The programme for the photographs of the corona during totality was agreed upon to give results comparable with those obtained by Professor Miller in Sumatra, at Alor Star and at other stations. Lieut. Blagden was in charge of this instrument, with the assistance of Capt. Sinnott and Lieut. Haynes, who also divided the task of counting seconds before and during the 5 minutes of totality.

5. A double tube polariscope of 6-foot focal length, with a large Nicol prism placed in front of one tube. The instrument has been used by Professor Newall at several eclipses. In order that observations might be made at points of the limb which were likely to be of special interest owing to the presence of prominences or other disturbances, it was arranged by Dr. Royds that a code telegram should be sent from Kodaikanal Observatory giving the latest possible information as to the limb of the Sun. The programme of the setting of the Nicol prism for the 8 possible exposures was only finally settled after the arrival of this wire. Professor Barnes was in charge of this instrument, assisted by Mr. Abel Smith. It was also arranged that observations should be made with a Nicol and Savart prism of the direction of polarisation of the sky light in the neighbourhood of the Sun.

6. Part of the beam from the 16-inch cœlostat feeding the polariscopic camera was diverted by a right-angled prism to feed a small Rowland concave grating used as an objective grating: with this it was proposed to photograph the spectra of the flash and the corona in the region about 7500 A. for which plates stained with kryptocyanin are sensitive. Professor Kibble was in charge of this instrument.

The climate of Pattani was very hot, and although it was known that ice was available it was decided to test a small refrigerating plant as a further aid in photographic development.

The Silicagel Company specially designed and kindly lent a unit to cool 10 gallons of water through 15° F. Unfortunately the unit was damaged in transit to Pattani and we were unable to test it fully, but the experiment was successful in showing how useful such a plant would be in an eclipse camp in a tropical country out of reach of an easy supply of ice.

In advance of the eclipse Dr. Royds undertook to make some experiments on sensitising photographic plates to the infra-red by making

some trials in Madras where the temperature and humidity would be similar to those at the 1929 eclipse site in Siam, and he sends the following report: "The Eastman Kodak Company of New York kindly supplied one-tenth milligram of neocyanin dye with instructions, and two dozen Kodak infra-red sensitive plates. Preliminary experiments were first made at Kodaikanal, which has a temperate climate, and methods were first worked out there before proceeding to Madras. The Kodak plates were not used until they were about three months old—that is, about the same age as those which would be available at the eclipse. Trials were first made with ordinary plates sensitised by bathing and then hyper-sensitised with ammonia, and afterwards with the Kodak infra-red plates. The procedure adopted was the same as that recommended by the Eastman Kodak Company, except that, after the ammonia bath for hyper-sensitising, the plates were rinsed for 30 seconds in methyl alcohol to ensure quick drying. The use of a drying cabinet was dispensed with as causing unnecessary loss of time when rapid drying was essential. It was found best to dry the plates in an open rack placed in the blast from an electric fan. In order to ascertain the times required for drying plates, blank experiments were first conducted with the dark room illuminated. It was found that the plates dried in less than 1 minute after the neocyanin bath and in about 3 minutes after the ammonia bath. Possibly the plates were not bone dry in these times, but they were sufficiently dry for fingers to leave no impression on the emulsion. In Madras, where the temperature was 90° F., humidity 80 per cent., and without the use of ice or electric fan these times became about 1½ minutes and 5 minutes respectively.

"With the Kodak commercial infra-red plates about three months old and then hyper-sensitised with ammonia, it was found that their speed to the calcium infra-red lines was about the same as that of ordinary plates freshly bathed in neocyanin and then hyper-sensitised, but the former were somewhat cleaner in working. Consequently, it was decided to use Kodak commercially prepared infra-red sensitive plates at the eclipse camp. The concentrated dye solution was also taken in case of failure of the commercial plates, but was never used at the eclipse camp. When the commercial plates were one year old, it was found that they were slightly slower than the bathed plates.

"It was of great assistance to find that Kodak neocyanin plates could be desensitised with Desensitol (Ilford Company), which enabled development of trial plates to be undertaken in white light in the dark room. Naturally, it was not proposed to desensitise the eclipse plates before development.

"For economy in the use of methyl alcohol, the same alcohol was used several times until the time required for drying became about 1½ times as long as that required for fresh alcohol. Alcohol previously used for drying hyper-sensitised plates was never used for drying bathed plates before hyper-sensitising.

"At the eclipse camp at Pattani, Siam, power for an electric fan in

the dark room was fortunately available through the kindness of the Siamese Government. The time allowed for drying the hyper-sensitised plates to be used at the eclipse was lengthened to 7 or 8 minutes as a precaution, and it was found that the physical conditions in the dark room were not too trying with the fan blowing.

"*Failures.*—No failure of sensitiveness to the infra-red occurred with hyper-sensitised Kodak commercial plates at any time. Occasionally failures did, however, occur in the sensitising of ordinary plates with previously used baths and alcohol. After using a bath four or five times it was found that the sensitising power for the infra-red appeared to be suddenly inhibited. When this occurred for the first time it was thought that the plate had not been properly treated owing to some bungling in the dark, and an attempt was made to sensitise another plate with the same baths. This, too, was found insensitive, and it was thought that on the former occasion the baths might have been poured back into the wrong bottles, so that success could not again be hoped for. The same effect, however, occurred on another occasion after the baths had been used several times, but it was difficult to be certain that some mistake might not again have been made in the darkness. The causes of failure were not further investigated, but thereafter the baths and alcohol were thrown away after using three or four times, and no further failures occurred. No failures ever occurred with freshly prepared baths and fresh alcohol."

The weather was hot and fine for the most part during the weeks preceding the eclipse. The Sun was clear at eclipse time six days out of seven during the seven weeks preceding the eclipse. Preparations were easily complete in time, but a spell of bad weather set in three days before the eclipse. Heavy rain fell the whole night before the eclipse day, and at the time of the eclipse clouds obscured the Sun. The diminishing crescent was glimpsed at times through the clouds, and during totality Mr. Melotte made two exposures with the Astrographic Telescope, on which the great prominence was shown—but no image of the crescent could be seen on the slit-jaws of the spectrographs. It had not been possible to carry out any of the proposed programme of observations.

No report of this expedition would be complete which omitted reference to the most generous and friendly assistance which was received on all sides from the Siamese authorities.

His Majesty the King of Siam appointed a reception committee to welcome the two astronomical expeditions sent to Siam, that of Professor Rosenberg of Kiel and the British expedition. This committee consisted of Rear-Admiral Phya Rajvangsen, A.D.C., President, Colonel Phra Salvidhan Nides, First Councillor Phya Srishdikar Barnchong, Second Councillor Phya Sri Sena, Mr. Brandli, Government Meteorologist, and Commander Phra Riem Virajbhakaya, Secretary. In addition to building a comfortable and well-equipped bungalow camp to house the expedition, the committee also paid all the expenses for labour in building the pillars and atap shelters, and provided electric lighting sets both for the eclipse camp—for electric arcs, dark-room, etc.—and

for the bungalow camp. Further, they and the local authorities saw to it that every facility possible was available to help the expedition. The Lord Lieutenant of Pattani, His Excellency Phya Udombongso Bhensvaadi, and the Governor of Pattani, Phra Vides Phattanatorn, assisted in every way, personally and through their staffs; the Lord Lieutenant put the services of his Secretary, Nai Chamlong Asavasena, freely at the disposal of the expedition as interpreter and helper in every way. A special tribute must be paid to his untiring efforts to help the expedition. The club room of the local troop of boy scouts, which adjoined the eclipse camp, was made available as a store for the apparatus, and the hut dark-room was set up inside it. Local police guarded the camp day and night without any charge on the funds of the expedition. Customs facilities for the entrance, duty free, of all the apparatus and personal luggage of members of the expedition were granted freely. And to show the thorough care with which the interests of the expedition were considered, it may be mentioned that a special doctor from Bangkok, Luang Jarn Vidli Bej, was stationed at Pattani while the expedition was there, and that he was supplied with anti-snakebite serum in case of an accident to any member of the expedition.

Their Majesties the King and Queen of Siam honoured the British expedition with three visits to the camp. On the day before the eclipse they inspected the instruments and watched a rehearsal of the eclipse programme; in the evening they returned to look at some celestial objects through the Astrographic Telescope and the Greenwich 4-inch portable telescope, an instrument used to cast an image of the Sun on a screen for their Majesties' use during the partial phase of the eclipse. On the day of the eclipse their Majesties, with a small suite, took up a position in a Royal pavilion inside the eclipse camp. It was a special disappointment to the members of the eclipse expedition that the cloudy sky prevented the Royal visitors, to whom they were so heavily indebted, from seeing the wonderful spectacle of the eclipsed Sun.

Finally reference must be made to the zealous way in which Col. Waley Cohen, the voluntary Secretary of the expedition, helped on every hand. By taking complete charge of the living camp, making all arrangements with the local authorities, and seeing to the necessary financial arrangements, he set the observers free of the detailed worries which are normal to camp life. His services were too varied to detail, but he gave unstinted assistance when asked and placed everybody under a debt of gratitude.

### (*d*) Observations made on Board H.M.S. "Iroquois."

On the morning of May 9, H.M.S. *Iroquois* (Capt. J. D. Nares, R.N.) proceeded from Penang and took up a position in Long. 99° 59′ 52″ E., Lat. 5° 54′ 15″ N., near the central line of the eclipse, and lay at anchor there during the whole of the eclipse. The times of the 2nd and 3rd

contacts were observed by Capt. Nares and two other officers, while Lieut. Foulerton, R.N., made a long series of meteorological observations. The conditions were quite similar to those at Alor Star, and a short summary of them may be of interest. The sky was covered throughout by high cloud, stratus, cumulus, and cirrus. At times the Sun was completely obscured by cumulus, but generally it was visible through thin cloud. At local standard noon (5 a.m. Greenwich), just before first contact, the record was :

Wet bulb 79½°. Dry bulb 86°. Barometer 1007·5 millibars.

Clouds alto stratus, cumulus, cumulo-nimbus. Clouds increasing, but Sun still fairly clear.

The following details for phenomena round about the time of totality are copied in full from the official report of Lieut. Foulerton. The times are $7^h$ fast on Greenwich.

13.30. Wet bulb 78½°. Dry bulb 84°.

Light and shadows very queer and unreal. Exceedingly dark to the N.W., with black clouds and general appearance of the approach of a very heavy Sumatra.

13.33. Wet bulb 78½°. Dry bulb 83¾°.

Start of totality. At first moment of totality the sky and clouds to the S.W. became a vivid emerald green which gradually changed into colour mentioned below. A faint solar corona, and, on the clouds near the Sun, all the spectrum colours. Through gaps in the clouds the sky showed a deep, dark blue. One planet, Jupiter, clearly visible.

13.38. Wet bulb 78½°. Dry bulb 83¾°.

End of totality. A crimson flame suddenly appeared at the trailing edge of the Moon ; through this shot a thin spear of white light which in an instant became like a small, but very brilliant, arc light, and giving similar lights and shadow—within 30 seconds the sky became quite light again.

During totality the general appearance of the sky was the same as will be seen between a quarter and half an hour before sunrise. The horizon was firm and colours—though flat elsewhere—appeared in the sky and on the clouds, to the westward, as they do in the period before the Sun rises, but very much accentuated—that is, all rather hard shades of blue, green, yellow, and pink. This was most striking.

Except for the crimson flame no colours were observed round the edge of the Moon—though spectrum colours were seen inside the corona, reflected on the clouds just previous to totality. The time of the Moon uncovering the Sun was far more definite than the first moment of totality.

13.40. Conditions returning to normal. The light appeared to return quicker than it faded away.

The total drop in the dry-bulb thermometer from the start of the eclipse was only 2¼°. The movement of the wet-bulb thermometer was less regular, and the total fall rather less. The variation of the barometer throughout the day was normal.

The Greenwich times of the observed second and third contacts (mean of three observers) were as follows :—

Second contact, $6^h\ 33^m\ 26^s{\cdot}85$.
Third contact, $6^h\ 38^m\ 17^s{\cdot}35$.

The computed times from the data given in the *Nautical Almanac* were respectively $0^s{\cdot}0$ and $1^s{\cdot}6$ later. Consequently the computed times were nearly correct. This result is supported by the appearance of the photographs taken near the beginning and end of totality with the 45-foot telescope at Alor Star.

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## *Stellar Parallaxes Determined at the Royal Observatory, Greenwich.*

(*Communicated by the Astronomer Royal.*)

The following list is a continuation of the results obtained with the 26-inch telescope, and published annually in the November number of the *Monthly Notices*.

In the column "Proper Motion" the first value given is from meridian observations ; the second is the relative proper motion derived from the photographs. The motions in both R.A. and Dec. are expressed in arc of a great circle.

In calculating the parallax the relative proper motion has been used, *not* the meridian proper motion.

In computing the P.E. an average value $\pm''{\cdot}025$ has been adopted as the probable error of a single plate of unit weight. This value has been derived from a discussion of the residuals of 333 determinations of parallax, each of which depends on 23 plates on the average. The P.E. has accordingly been calculated by the formula

$$r = \frac{\pm''{\cdot}025}{\sqrt{w}}.$$

The first P.E. given is that of the parallax, the second is the P.E. of the relative proper motion.

Generally six comparison stars of mean magnitude 11–12 have been used, of which good images are obtained with an exposure of three minutes. The magnitude of the parallax star has been reduced to equality by means of the rotating sector and (for the brighter stars) local desensitisation by means of a weak solution of copper sulphate.

The Visual Magnitude is taken from the *Bonn Durchmusterung*, the Photographic from the *Greenwich Astrographic Catalogue*.

| No. | Name and B.D. No. | Mag. Vis. Photog. | Spectral Type. | R.A. Dec. 1900·0. | Proper Motion. R.A. | Proper Motion. Dec. | Parallax. | P.E. | No. of Plates. |
|---|---|---|---|---|---|---|---|---|---|
| | | | | h m s | ″ | ″ | ″ | ″ | |
| 1 | 76° 39 | 7·0 | K2 | 1 11 6 | −·088 | −·029 | +·004 | ±·007 | 23 |
| | | 8·2 | | +76° 16′ | −·077 | | | ±·008 | |
| 2 | P. III 102 | 6·2 | F2 | 3 36 33 | +·097 | −·102 | ·000 | ±·007 | 20 |
| | 66° 284 | 6·1 | | +66° 53′ | +·090 | | | ±·008 | |
| 3 | Gr. 774 | 5·0 | G5 | 4 7 59 | −·050 | +·087 | +·010 | ±·006 | 25 |
| | 82° 113 | 6·1 | | +83° 6′ | −·049 | | | ±·004 | |
| 4 | O.A. 5237 | 8·5 | Ko | 4 46 53 | +·100 | −·009 | +·012 | ±·006 | 27 |
| | 64° 487 | 9·3 | | +64° 14′ | +·047 | | | ±·003 | |
| 5 | R. 1476 | 7·5 | Go | 5 24 24 | −·030 | −·170 | +·021 | ±·005 | 30 |
| | 74° 250 | 8·1 | | +74° 37′ | −·035 | | | ±·004 | |
| 6 | 69° 361 | 8·1 | G5 | 5 55 4 | −·154 | −·017 | +·031 | ±·006 | 27 |
| | | 8·5 | | +69° 29′ | −·151 | | | ±·004 | |
| 7 | Anon | .. | | 6 58 35 | +·026 | −·258 | −·012 | ±·007 | 17 |
| | | 12·8 | | +64° 29′ | +·047 | | | ±·006 | |
| 8 | A.C. 64° 2738 | .. | | 7 48 36 | +·043 | −·177 | +·004 | ±·006 | 28 |
| | | 12·0 | | +64° 49′ | +·062 | | | ±·004 | |
| 9 | A.C. 71° 4803 | .. | | 8 50 43 | +·094 | −·498 | +·018 | ±·006 | 25 |
| | | 10·7 | | +71° 4′ | +·143 | | | ±·005 | |
| 10 | $\sigma^2$ UMa. | 5·3 | F8 | 9 1 36 | −·005 | −·070 | +·075 | ±·007 | 24 |
| | 67° 577 | 5·4 | | +67° 32′ | +·003 | | | ±·004 | |
| 11 | 78° 372 | 9·5 | | 10 59 15 | −·195 | −·064 | +·021 | ±·008 | 24 |
| | | 10·7 | | +78° 38′ | −·164 | | | ±·006 | |
| 12 | Gr. 1852 | 5·8 | Ko | 12 0 10 | +·144 | −·089 | +·008 | ±·007 | 24 |
| | 77° 461 | 6·7 | | +77° 27′ | +·155 | | | ±·008 | |
| 13 | $\kappa$ Dra. | 3·3 | B5p | 12 29 13 | −·060 | +·006 | +·021 | ±·008 | 21 |
| | 70° 703 | 3·6 | | +70° 20′ | −·053 | | | ±·008 | |
| 14 | Gr. 2007 | 7·0 | Go | 13 18 39 | −·108 | +·023 | +·052 | ±·009 | 21 |
| | 85° 222 | 7·5 | | −85° 16′ | −·137 | | | ±·007 | |
| 15 | 4 UMi. | 5·0 | Ko | 14 9 14 | −·028 | +·026 | −·015 | ±·008 | 24 |
| | 78° 478 | 6·1 | | +78° 1′ | −·031 | | | ±·008 | |

| No. | Name and B.D. No. | Mag. Vis. Photog. | Spectral Type. | R.A. Dec. 1900·0. h m s | Proper Motion. R.A. | Proper Motion. Dec. | Parallax. | P.E. | No. of Plates. |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | ″ | ″ | ″ | ″ | |
| 16 | $\beta$ UMi. | 2·1 | K5 | 14 51 0 | −·030 | +·005 | +·018 | ±·007 | 29 |
| | 74° 595 | 3·8 | | +74° 33′ | −·013 | | | ±·004 | |
| 17 | C. 2382 | 8·9 | Ko | 15 43 6 | +·016 | +·033 | +·012 | ±·010 | 20 |
| | 85° 264 | 9·2 | | +85° 1′ | +·027 | | | ±·009 | |
| 18 | L.F. 2832 | 7·0 | Go | 16 37 45 | +·118 | −·082 | +·067 | ±·009 | 23 |
| | 80° 519 | 7·3 | | +79° 59′ | +·077 | | | ±·007 | |
| 19 | $\varepsilon$ UMi. | 4·0 | Go | 16 56 12 | +·003 | +·015 | +·012 | ±·008 | 28 |
| | 82° 498 | 5·3 | | +82° 12′ | +·023 | | | ±·008 | |
| 20 | C. 2591 | 8·3 | Ko | 17 13 45 | −·013 | +·074 | +·026 | ±·009 | 23 |
| | 81° 574 | 9·2 | | +81° 30′ | −·038 | | | ±·008 | |
| 21 | O.A. 19158 | 8·5 | Ko | 19 15 28 | +·035 | +·132 | +·027 | ±·009 | 27 |
| | 66° 1181 | 8·7 | | +66° 5′ | +·004 | | | ±·007 | |
| 22 | $\beta$ Cep. | 3·0 | B1 | 21 27 23 | +·011 | +·005 | +·006 | ±·009 | 25 |
| | 69° 1173 | 7·8<br>2·4 | | +70° 7′ | −·003 | | | ±·007 | |
| 23 | C. 3351 | (8·3) | Ko | 21 47 37 | +·155 | +·064 | +·005 | ±·009 | 21 |
| | 86° 325 | 9·3 | | +86° 24′ | +·145 | | | ±·008 | |
| 24 | C. 3352 | (8·3) | Ko | 21 47 42 | +·153 | +·070 | −·003 | ±·009 | 20 |
| | 86° 325 | 9·7 | | +86° 24′ | +·117 | | | ±·008 | |
| 25 | $\xi$ Cep. | 4·5 | A3 | 22 0 53 | +·215 | +·086 | +·021 | ±·008 | 27 |
| | 63° 1802 | 6·5<br>4·7 | | +64° 8′ | +·179 | | | ±·005 | |
| 26 | RZ Cep. | 9·2 | | 22 35 43 | +·070 | +·170 | +·005 | ±·008 | 26 |
| | 64° 1694 | Var. | | +64° 20′ | +·114 | | | ±·006 | |

No. 23.—The visual magnitude and the spectral type refer to the combined mass.

Nos. 22, 25.—Brighter component observed.

No. 26.—Limits of photographic magnitude $9^{m}\cdot0$ and $9^{m}\cdot9$.

## *On the Radial Limitation of the Sun's Magnetic Field.*

By T. G. Cowling, B.A.

(*Communicated by Prof. E. A. Milne.*)

1. Recently the problem of the radial limitation of the Sun's magnetic field has been attracting the attention of workers in astrophysics. Papers have been published by Ross Gunn * and Professor S. Chapman,† in which attempts are made to explain this phenomenon. The methods of both are based on a consideration of the motions of the charged particles that exist in the solar atmosphere, under the action of magnetic, electric, and gravitational forces.

Chapman's method is as follows. Under the action of a constant magnetic field and some other field of force a charged particle describes a spiral path. The mean motion of particles of a given type in their free paths has a small component perpendicular to both the magnetic and mechanical forces. This component is oppositely directed for particles with charges of opposite signs; hence an electric current is set up which reduces the magnetic field very rapidly as we go outward along the radius.

Ross Gunn's method is based on an idea which has been used by Schrödinger ‡ and others to explain diamagnetism in metals. Schrödinger assumed that the metallic atoms are fixed, but that free electrons are free to move under the applied magnetic force. They describe circular arcs in their free paths; this produces a diamagnetic effect, as the electrons describing circles are equivalent to magnetic doublets whose axes are in a direction opposed to the applied magnetic force. Ross Gunn applied this method to both ions and electrons in the solar atmosphere and found that diamagnetism should appear, which would reduce the field in the required manner. He also noticed in his third paper that another effect appears, as the magnetic field is not uniform. Under the action of this magnetic field, charged bodies no longer describe circles about the lines of force, but undergo a motion in which to a circular motion is added a slight drift across the lines of force. An electric current is thus set up which is in a direction opposed to that which Chapman demands. The effect is to oppose the limitation of the magnetic field.

It is clear that some method of attack should be possible by which all these effects are combined in the one treatment. We shall try to supply that here. In so doing, we find that effects arise which make the theories of the limitation fail. Before so doing we shall adduce certain considerations that will make it seem probable that this should occur.

2. Let us consider first the limitation theory of Ross Gunn. When Schrödinger used this method to explain diamagnetism, it was pointed out by others § that in certain circumstances the boundary effect

* *Phys. Rev.*, **32**, 133 (1928); **33**, 614, 832 (1929); **34**, 335 (1929); *Astrophysical Journ.*, **69**, 287 (1929).

† *M.N.*, **89**, 59 (1928).

‡ *Wien Sitzungsberichte*, **121**, 1305 (1912).

§ See van Leeuwen, *Journ. de Phys.*, **2**, 374 (1921).

will mask the interior effect and reduce the apparent diamagnetism to zero. To understand this, let us consider a piece of metal in a uniform magnetic field. As Schrödinger shows in Appendix II. of his paper, the mean velocity of electrons at any interior point of the metal is zero, and the velocity distribution is Maxwellian. Hence it is only if the boundary is such that a Maxwell distribution cannot exist up to the boundary that we have any effect. The effect is due to superficial currents. If a Maxwell distribution can exist up to the boundary, *e.g.* if the boundary can be regarded as sharp and such that electrons perform elastic impacts on it, the diamagnetic effect is lost.

This analysis can be taken over to the solar problem. The diamagnetic effect is a boundary effect: if there is no boundary, as is the case in this problem, the effect seems to be lost. There only remains the possibility that the decrease of density may give rise to an edge effect; we shall show later that this is not so.

We do not discuss Ross Gunn's second idea here: for details we refer to the original paper. He does not regard it as of importance in the limitation theory. We shall show later that the effect appears to vanish completely, as well as Chapman's effect, on taking account of all the factors present.

3. We will now consider Chapman's work in some detail. He introduces other factors than those considered by Gunn in that he considers the effect of gravitational and electric forces.

We consider only the problem as it presents itself at the equator. Here the magnetic force H is horizontal and from south to north. Take the axis O$z$ in this direction; let O$x$ be vertically upward and O$y$ from east to west. We neglect curvature and suppose that the mass of gas is isothermal.

Consider the motion of a particle of mass $m$ and charge $e$ in E.S.U. under the action of the magnetic force H and a constant force X parallel to O$x$. Its equations of motion are

$$m\ddot{x} = \mathrm{X} + \frac{e\mathrm{H}}{c}\dot{y}, \qquad m\ddot{y} = -\frac{e\mathrm{H}}{c}\dot{x}, \qquad m\ddot{z} = 0 \quad . \qquad (1)$$

If the initial values of $\dot{x}$, $\dot{y}$, $\dot{z}$ are denoted by $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ and if $\omega = e\mathrm{H}/mc$, the values of $\dot{x}$, $\dot{y}$, $\dot{z}$ after time $t$ are given by

$$\left.\begin{aligned} \dot{x} &= \dot{x}_0 \cos\omega t + \left(\dot{y}_0 + \frac{\mathrm{X}c}{e\mathrm{H}}\right)\sin\omega t \\ \dot{y} &= -\frac{\mathrm{X}c}{e\mathrm{H}}(1 - \cos\omega t) + \dot{y}_0 \cos\omega t - \dot{x}_0 \sin\omega t \\ \dot{z} &= \dot{z}_0 \end{aligned}\right\} \quad . \qquad (2)$$

Thus if we suppose that the mean values of $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ are equal to zero, and average with respect to the time, we get for the mean values of $\dot{x}$, $\dot{y}$, $\dot{z}$ the values

$$0, \quad -\frac{\mathrm{X}c}{e\mathrm{H}}, \quad \text{and} \quad 0. \ .$$

Thus there is apparently a drift velocity $-\frac{Xc}{eH}$ of such particles parallel to $Oy$. Actually the drift is smaller if we consider collisions. Let $\tau$ be the mean interval between collisions. Then at any instant we may take the fraction of atoms which commenced their free paths between times $t$ and $t+dt$ ago as

$$\frac{1}{\tau}e^{-t/\tau}dt.$$

These have a mean drift velocity equal to $-\frac{Xc}{eH}(1-\cos\omega t)$. Hence the mean drift velocity at any instant is

$$V = -\frac{Xc}{eH}\int_0^\infty (1-\cos\omega t)e^{-t/\tau}\frac{dt}{\tau}$$

$$= -\frac{Xc}{eH}\frac{\tau^2\omega^2}{1+\tau^2\omega^2} \quad . \quad . \quad . \quad . \quad (3)$$

Let us apply this to ions and to electrons, of charges $e$ and $-e$. We denote by suffixes $\iota$ and $e$ that the quantities in question relate to ions or electrons. If the acceleration of gravity is $g$ and the electric force is E along $Ox$, then

$$X_\iota = -m_\iota g + eE, \qquad X_e = -m_e g - eE.$$

The Pannekoek-Rosseland theory of the electric field of a star states that

$$X_\iota = X_e$$

$$\therefore \quad eE = \tfrac{1}{2}(m_\iota - m_e)g$$

$$\therefore \quad X_\iota = X_e = -\tfrac{1}{2}(m_\iota + m_e)g \quad . \quad . \quad . \quad (4)$$

Now the electric current density parallel to $Oy$ is

$$i = (n_\iota eV_\iota - n_e eV_e)/c,$$

where $n_\iota$, $n_e$ are the numbers of ions and electrons per unit volume. Substituting the values of $V_\iota$ and $V_e$ from (3), and using the values of $X_\iota$, $X_e$ given by (4), we have

$$i = \frac{(m_\iota + m_e)g}{2H}\left\{\frac{\tau_\iota^2\omega_\iota^2 n_\iota}{1+\tau_\iota^2\omega_\iota^2} + \frac{\tau_e^2\omega_e^2 n_e}{1+\tau_e^2\omega_e^2}\right\}.$$

Using the known values of $\tau_\iota$, $\tau_e$, $\omega_\iota$, $\omega_e$ we obtain

$$i = \frac{(m_\iota + m_e)g}{2H}\left\{\frac{n_\iota}{1+\beta_\iota\frac{n_\iota^2}{H^2}} + \frac{n_e}{1+\beta_e\frac{n_\iota^2}{H^2}}\right\} \quad . \quad . \quad (5)$$

where $\beta_\iota$, $\beta_e$ are constants depending on the physical constants of the ions and electrons, and on the temperature.

Now we have, as Chapman showed,

$$i = -\frac{1}{4\pi}\frac{dH}{dx}.$$

Also, if $p_\iota$, $p_e$ are the partial pressures of ions and electrons,

$$\frac{dp_\iota}{dx} = n_\iota X_\iota, \qquad \frac{dp_e}{dx} = n_e X_e \quad . \quad . \quad . \quad (6)$$

Hence

$$\frac{dp_\iota}{dx} = -\tfrac{1}{2}(m_\iota + m_e)gn_\iota,$$

$$\frac{dp_e}{dx} = -\tfrac{1}{2}(m_\iota + m_e)gn_e;$$

hence

$$\frac{1}{4\pi}H\frac{dH}{dx} = \left[\frac{\frac{dp_\iota}{dx}}{1+\beta_\iota\frac{n_\iota^2}{H^2}} + \frac{\frac{dp_e}{dx}}{1+\beta_e\frac{n_\iota^2}{H^2}}\right].$$

If we suppose that $p_\iota = p_e$ (as is approximately true) and that K is constant, where

$$K = \left(1+\beta_\iota\frac{n_\iota^2}{H^2}\right)^{-1} + \left(1+\beta_e\frac{n_\iota^2}{H^2}\right)^{-1},$$

we have

$$\frac{1}{4\pi}H\frac{dH}{dx} = K\frac{dp_\iota}{dx},$$

and so

$$\frac{1}{8\pi}H^2 = Kp_\iota + \text{const.} \quad . \quad . \quad . \quad (7)$$

This concludes our brief account of Chapman's work.

4.* The above investigation needs correction at two points. It assumes that the magnetic field has no effect on the electric field on the one side, or on the mechanical equilibrium on the other.† We must try to rectify this. Let us deal first with the macroscopic equilibrium. In addition to the force $X_\iota$ per ion, there will be on the ions contained in an element of volume a mean force due to the motion in a magnetic field, of magnitude $eV_\iota H/c$ per ion, along $Ox$. Thus, using the value of $V_\iota$ from (3),

$$\frac{dp_\iota}{dx} = n_\iota\left(X_\iota + eH\frac{V_\iota}{c}\right)$$

$$= n_\iota X_\iota\left(1 - \frac{\tau_\iota^2\omega_\iota^2}{1+\tau_\iota^2\omega_\iota^2}\right) = X_\iota\frac{\beta_\iota n_\iota^3}{H^2+\beta_\iota n_\iota^2} \quad . \quad (8)$$

* For the ideas in this and the following two sections I am largely indebted to Professor Milne.

† The mechanical effect of the magnetic field has been recently considered by Professor Chapman, but his paper is not yet published. See *Observatory*, **52**, 193, 1929.

and similarly

$$\frac{dp_e}{dx} = n_e X_e\left(1 - \frac{\tau_e^2\omega_e^2}{1+\tau_e^2\omega_e^2}\right) = X_e\frac{\beta_e n_\iota^2 n_e}{H^2+\beta_e n_\iota^2} \quad . \quad (8')$$

Formulæ (8) and (8′) are the modified forms of equations (6). For neutral atoms we similarly get an equation

$$\frac{dp}{dx} = nX = -n(m_\iota + m_e)g \quad . \quad . \quad . \quad (8'')$$

where $n$ is the number of neutral atoms per unit volume, $p$ is their partial pressure, and X the force on one of them.

We have another equation, as yet unused, which must be satisfied. This is the equation of dissociative equilibrium,

$$p_\iota p_e/p = \theta,$$

where $\theta$ depends on temperature alone. As we are considering an isothermal atmosphere, this gives

$$\frac{1}{p}\frac{dp}{dx} = \frac{1}{p_\iota}\frac{dp_\iota}{dx} + \frac{1}{p_e}\frac{dp_e}{dx}$$

or

$$\frac{1}{n}\frac{dp}{dx} = \frac{1}{n_\iota}\frac{dp_\iota}{dx} + \frac{1}{n_e}\frac{dp_e}{dx} \quad . \quad . \quad . \quad (9)$$

Comparing (9) with (8), (8′), and (8″) and using the known values of $X_\iota$, $X_e$, and X, we find

$$(-m_\iota g + e\mathrm{E})\frac{\beta_\iota n_\iota^2}{H^2+\beta_\iota n_\iota^2} + (-m_e g - e\mathrm{E})\frac{\beta_e n_\iota^2}{H^2+\beta_e n_\iota^2} = -(m_\iota + m_e)g.$$

$$\therefore\ g\left\{\frac{H^2 m_\iota}{H^2+\beta_\iota n_\iota^2} + \frac{H^2 m_e}{H^2+\beta_e n_\iota^2}\right\} = e\mathrm{E}\left\{\frac{H^2}{H^2+\beta_\iota n_\iota^2} - \frac{H^2}{H^2+\beta_\iota n_\iota^2}\right\}.$$

This gives a value of $e\mathrm{E}/g$ if $H \neq 0$. Let $H \to 0$. Then $e\mathrm{E}/g$ tends to the limit

$$\left(\frac{m_\iota}{\beta_\iota} + \frac{m_e}{\beta_e}\right)\Big/\left(\frac{1}{\beta_\iota} - \frac{1}{\beta_e}\right).$$

It is easy to see that this cannot be equal to $\frac{1}{2}(m_\iota - m_e)$ ; if it were, we should get

$$(m_\iota + m_e)\left(\frac{1}{\beta_\iota} + \frac{1}{\beta_e}\right) = 0,$$

which is not so. Hence in the limit we do not get the Pannekoek-Rosseland field, as we should. Thus we appear to have reached a contradiction.

5. We can also reach the same contradiction as follows. The Pannekoek-Rosseland field is determined by supposing that $n_\iota/n_e$ is

constant. If we suppose that the electric field, modified by the presence of the magnetic field, is still such that $n_\iota/n_e$ is constant, then

$$\frac{1}{n_\iota}\frac{dn_\iota}{dx} = \frac{1}{n_e}\frac{dn_e}{dx}$$

or

$$\frac{1}{n_\iota}\frac{dp_\iota}{dx} = \frac{1}{n_e}\frac{dp_e}{dx}.$$

Using equations (8) and (8′), we obtain the result

$$\frac{(-m_\iota g + e\mathrm{E})\beta_\iota n_\iota^2}{\mathrm{H}^2 + \beta_\iota n_\iota^2} = \frac{(-m_e g - e\mathrm{E})\beta_e n_\iota^2}{\mathrm{H}^2 + \beta_e n_\iota^2}.$$

$$\therefore\ g(m_\iota\beta_\iota(\mathrm{H}^2 + \beta_e n_\iota^2) - m_e\beta_e(\mathrm{H}^2 + \beta_\iota n_\iota^2)) = e\mathrm{E}\{\mathrm{H}^2(\beta_\iota + \beta_e) + 2\beta_\iota\beta_e n_\iota^2\}.$$

Using this value of $e$E in (8) and (8′), we have

$$\frac{dp_\iota}{dx} = \frac{-\beta_e\beta_\iota n_\iota^3(m_\iota + m_e)g}{\mathrm{H}^2(\beta_\iota + \beta_e) + 2\beta_\iota\beta_e n_\iota^2} \quad . \quad . \quad . \quad (10)$$

$$\frac{dp_e}{dx} = -\frac{\beta_\iota\beta_e n_\iota^2 n_e(m_\iota + m_e)g}{\mathrm{H}^2(\beta_\iota + \beta_e) + 2\beta_\iota\beta_e n_\iota^2} \quad . \quad . \quad . \quad (10')$$

Substituting the values of $\frac{dp}{dx}$, $\frac{dp_\iota}{dx}$, $\frac{dp_e}{dx}$ from (8″), (10), and (10′) into the equation of dissociative equilibrium (9), we obtain

$$\frac{2\beta_\iota\beta_e n_\iota^2(m_\iota + m_e)g}{\mathrm{H}^2(\beta_\iota + \beta_e) + 2\beta_\iota\beta_e n_\iota^2} = (m_\iota + m_e)g,$$

which can only hold if H = o. Hence we have again reached a contradiction.

6. The second of the deductions we have given has a fundamental assumption, which, though probable, may be unsound, viz. that $n_\iota/n_e$ is constant. This is not true of the first. The only point of that which is open to criticism is one common to both, viz. the introduction of the law of dissociative equilibrium. As the temperature of the sun is not uniform, the validity of this step, and in particular of equation (9), seems in doubt. But we can produce an artificial problem in which conditions are isothermal and the same contradiction arises. Let us consider a mass of gas in an enclosed space in the form of an anchor-ring; let it be at a constant temperature; let there be imposed a magnetic force along its axis and electric and gravitational fields symmetrical with respect to this axis; then at interior points Chapman's equations are satisfied. If in this case the law of dissociative equilibrium is not satisfied, then at one point of the gas combination of ions and electrons must occur, and at another point dissociation of atoms occurs. Thus the first point becomes

colder and the second hotter; which contradicts the second law of thermodynamics.

This example not only suggests that there is something wrong with Chapman's theory, but it also suggests what is wrong. In the above we limited ourselves to the consideration of what happened in the interior of the anchor-ring. There might have been a boundary effect altogether different; and if we had considered a different way of finding the average velocity of an ion, this might have affected the interior results. We shall show later that there is in fact an error in Chapman's determination of the mean value of $\dot{y}$ from (2) in that the mean values of $\dot{x}_0$ and $\dot{y}_0$ may not be zero; this is, we shall discover, due to the pressure gradient.

7. Let us consider, then, the following problem. We suppose that there is above the bounding plane $x = 0$ an infinite mass of gas stratified in plane layers parallel to this plane. Let the directions of electric and gravitational force be parallel to $Ox$ and that of magnetic force be parallel to $Oz$. We wish to find the distributions of velocities of ions, electrons, and atoms in a steady state. The notation is the same as that used in § 3.

Let $f(u,v,w,x)\,du\,dv\,dw\,dx\,dy\,dz$ be the number of neutral atoms in the element of volume $dx\,dy\,dz$ at $(x, y, z)$ with component velocities between $u$ and $u + du$, $v$ and $v + dv$, $w$ and $w + dw$; and let $f_\iota$ and $f_e$ be similarly defined for ions and electrons. We wish to determine such a steady state as can exist under the circumstances. From what Chapman has proved it seems unlikely that a state macroscopically at rest can exist; but we shall postulate that there is no mass velocity parallel to $Ox$ or $Oz$, but only parallel to $Oy$, due to the drift velocities.

Consider the change in the functions $f$, $f_\iota$, $f_e$ in a small time $dt$. There are two factors working to modify them, collisions and diffusion. Consider the latter first. The atoms which at the end of the time $dt$ lie in the phase-element $dx\,dy\,dz\,du\,dv\,dw$ were, at the beginning of that time, in an equal element at $(x - udt,\ y - vdt,\ z - wdt)$ with velocities $(u + gdt,\ v,\ w)$. Hence the change in $f$ from diffusion is

$$-\left(\frac{\partial f}{\partial x}\,.\,u - \frac{\partial f}{\partial u}\,.\,g\right)dt.$$

Similarly the changes in $f_\iota$ and $f_e$ are

$$-\left\{\frac{\partial f_\iota}{\partial x}\,.\,u + \frac{\partial f_\iota}{\partial u}\left(-g + \frac{e\mathrm{E}}{m_\iota} + \frac{e\mathrm{H}}{m_\iota c}v\right) - \frac{\partial f_\iota}{\partial v}\,\frac{e\mathrm{H}}{m_\iota c}u\right\}dt$$

and

$$-\left\{\frac{\partial f_e}{\partial x}\,.\,u + \frac{\partial f_e}{\partial u}\left(-g - \frac{e\mathrm{E}}{m_e} - \frac{e\mathrm{H}}{m_e c}v\right) + \frac{\partial f_e}{\partial v}\,\frac{e\mathrm{H}}{m_e c}u\right\}dt.$$

Now let us consider the effect of collisions. The effect on $f$ is given by

$$-dt\int\int\int\int\int\sum_a (f\,.\,f_a' - \bar{f}\,.\,\bar{f}_a')\sigma_\sigma{}^2\mathrm{V}\cos\theta d\omega\,du'\,dv'\,dw',$$

where $f', \bar{f}, \bar{f}'$ denote $f(u', v', w'), f(\bar{u}, \bar{v}, \bar{w})$, and $f(\bar{u}', \bar{v}', \bar{w}')$ and $f_a$ denotes in turn $f, f_\iota$, and $f_e$. By $(\bar{u}, \bar{v}, \bar{w})$ and $(\bar{u}', \bar{v}', \bar{w}')$ are meant the velocities of those particles which collided with velocities $(u, v, w)$ and $(u', v', w')$. V is their relative velocity at collision, $\sigma_a$ the distance between their centres at collision, and $\theta$ the angle which the line of centres makes with the relative velocity ; while $d\omega$ is the solid angle in which the line of centres lies. Thus

$$\frac{\partial f}{\partial t} = -\frac{\partial f}{\partial x} \cdot u + \frac{\partial f}{\partial u} \cdot g$$

$$- \int\int\int\int\int \sum_a (f \cdot f_a' - \bar{f} \cdot \bar{f}_a') \sigma_a^2 \mathrm{V} \cos\theta d\omega \, du' \, dv' \, dw' \qquad (11)$$

and similarly for $\frac{\partial f_\iota}{\partial t}, \frac{\partial f_e}{\partial t}$. Actually

$$\frac{\partial f}{\partial t} = \frac{\partial f_\iota}{\partial t} = \frac{\partial f_e}{\partial t} = 0$$

as we have a steady state. Let us, however, consider the expressions

$$\mathrm{K}_\beta = \int\int\int f_\beta \log f_\beta du \, dv \, dw,$$

where $f_\beta$ is in turn $f, f_\iota$, and $f_e$. Then

$$\frac{\partial \mathrm{K}_\beta}{\partial t} = \int\int\int (1 + \log f_\beta) \frac{\partial f_\beta}{\partial t} du \, dv \, dw = 0.$$

Using the values obtained for $\frac{\partial f}{\partial t}, \frac{\partial f_\iota}{\partial t}$, and $\frac{\partial f_e}{\partial t}$, we have that $\frac{\partial \mathrm{K}_\beta}{\partial t}$ has two parts, corresponding to diffusion and collisions respectively. Consider the part due to diffusion. In $\frac{\partial \mathrm{K}}{\partial t}$ this is equal to

$$- \int\int\int (1 + \log f) \left( \frac{\partial f}{\partial x} \cdot u - \frac{\partial f}{\partial u} \cdot g \right) du \, dv \, dw$$

$$= -\frac{\partial}{\partial x} \int\int\int f \cdot \log f \cdot u \, du \, dv \, dw$$

$$+ g \int\int \Big[ f \log f \Big]_{u=-\infty}^{u=+\infty} dv \, dw.$$

The first term vanishes as there is no mass motion parallel to $Ox$; the second as $f \log f \to 0$ as $w \to \pm\infty$. Thus for $\frac{d\mathrm{K}}{dt}$ the diffusion terms vanish. Similarly the diffusion terms in $\frac{\partial \mathrm{K}_\iota}{\partial t}$ are

$$-\iiint(1+\log f_\iota)\left[\frac{\partial f_\iota}{\partial x}.u-\frac{\partial f_\iota}{\partial u}\left(g-\frac{e\mathrm{E}}{m_\iota}-\frac{e\mathrm{H}v}{m_\iota c}\right)-\frac{\partial f_\iota}{\partial v}\frac{e\mathrm{H}}{m_\iota c}u\right]du\,dv\,dw$$

$$=-\frac{\partial}{\partial x}\iiint f_\iota\log f_\iota\,u\,du\,dv\,dw$$

$$+\left(g-\frac{e\mathrm{E}}{m_\iota}\right)\iint\left[f_\iota\log f_\iota\right]_{u=-\infty}^{u=\infty}dv\,dw$$

$$-\frac{e\mathrm{H}}{m_\iota c}\iint\left[vf_\iota\log f_\iota\right]_{u=-\infty}^{u=\infty}dv\,dw$$

$$+\frac{e\mathrm{H}}{m_\iota c}\iint\left[uf_\iota\log f_\iota\right]_{v=-\infty}^{v=+\infty}du\,dw=0,$$

and those in $\dfrac{\partial \mathrm{K}_e}{\partial t}$ likewise vanish. Thus the condition

$$\frac{\partial \mathrm{K}}{\partial t}+\frac{\partial \mathrm{K}_\iota}{\partial t}+\frac{\partial \mathrm{K}_e}{\partial t}=0$$

is the same as if there were no forces acting. But this is the condition which gives us the velocity distributions in that case; hence we have the same form for the velocity distributions, viz.

$$f=\mathrm{A}\exp\{-\tfrac{1}{2}hm[(u-u_0)^2+(v-v_0)^2+(w-w_0)^2]\},$$

$$f_\iota=\mathrm{A}_\iota\exp\{-\tfrac{1}{2}hm_\iota[(u-u_0)^2+(v-v_0)^2+(w-w_0)^2]\},$$

$$f_e=\mathrm{A}_e\exp\{-\tfrac{1}{2}hm_e[(u-u_0)^2+(v-v_0)^2+(w-w_0)^2]\},$$

where $u_0$, $v_0$, $w_0$, A, $\mathrm{A}_\iota$, $\mathrm{A}_e$, $h$ may be functions of $x$, $y$, $z$. As we are considering a state in which there is no mass motion parallel to $Ox$ or $Oz$, $u_0=w_0=0$; also, as the gas is stratified in layers parallel to the plane $x=0$, A, $\mathrm{A}_\iota$, $\mathrm{A}_e$, $h$, $v_0$ can depend only on $x$. To determine the nature of this dependence on $x$, we return to our equations

$$\frac{\partial f}{\partial t}=\frac{\partial f_\iota}{\partial t}=\frac{\partial f_e}{\partial t}=0.$$

The collision terms in these equations now vanish identically; for the diffusion terms to vanish we must have that for all values of $u$, $v$, $w$

$$\frac{\partial f}{\partial x}.u-\frac{\partial f}{\partial u}.g=0,$$

$$\frac{\partial f_\iota}{\partial x}.u-\frac{\partial f_\iota}{\partial u}\left(g-\frac{e\mathrm{E}}{m_\iota}-\frac{e\mathrm{H}v}{m_\iota c}\right)-\frac{\partial f_\iota}{\partial v}\frac{e\mathrm{H}u}{m_\iota c}=0,$$

$$\frac{\partial f_e}{\partial x}.u-\frac{\partial f_e}{\partial u}\left(g+\frac{e\mathrm{E}}{m_e}+\frac{e\mathrm{H}v}{m_e c}\right)+\frac{\partial f_e}{\partial v}\frac{e\mathrm{H}u}{m_e c}=0.$$

These give that $h$, $v_0$ are constant and that

$$\begin{aligned} A &= C \exp(-hm\phi), \\ A_\iota &= C_\iota \exp\left(-hm_\iota\phi - he\psi - \frac{hev_0}{c}\chi\right), \\ A_e &= C_e \exp\left(-hm_e\phi + he\psi + \frac{hev_0}{c}\chi\right), \end{aligned}$$

where C, $C_\iota$, $C_e$ are constants, $\phi$, $\psi$ are the gravitational and electrostatic potentials, and $x = \int H dx$. This gives for the values of $f$, $f_\iota$, $f_e$

$$\left.\begin{aligned} f &= C \exp\{-hm\phi - \tfrac{1}{2}hm(u^2 + (v - v_0)^2 + w^2)\} \\ f_\iota &= C_\iota \exp\left\{-hm_\iota\phi - he\psi - \frac{hev_0}{c}\chi - \tfrac{1}{2}hm_\iota(u^2 + (v - v_0)^2 + w^2)\right\} \\ f_e &= C_e \exp\left\{-hm_e\phi + he\psi + \frac{hev_0}{c}\chi - \tfrac{1}{2}hm_e(u^2 + (v - v_0)^2 + w^2)\right\} \end{aligned}\right\} \quad (12)$$

These results are a generalisation of Schrödinger's result in Appendix II. of his paper.

8. We must now consider the effect of the bounding plane at $x = 0$. This was introduced to prevent the possibility that $g$ should be infinite. If we suppose the plane to be smooth and such that atoms, ions, and electrons perform elastic collisions with it, the forms of $f$, $f_\iota$, and $f_e$ will be unaffected by its presence.

Hence, in particular, corresponding to $v_0 = 0$, there can exist a stationary Maxwell distribution, in spite of the fact that such ions or electrons as do not strike the plane $x = 0$ have a mean drift parallel to $Oy$. It is easy to show that such ions or electrons as do strike the plane $x = 0$ have a mean drift opposed to this drift. Hence the total momentum of ions can be zero.

We have still to explain how a similar effect can occur in the upper layers. To see this, let us return to the equations (2). These can be written

$$\dot{x} = \alpha \cos(\omega t + \epsilon), \qquad \dot{y} = -\frac{cX}{eH} - \alpha \sin(\omega t + \epsilon), \qquad \dot{z} = \beta \quad (13)$$

where

$$\alpha = +\left[\dot{x}_0^2 + \left(\dot{y}_0 + \frac{Xc}{eH}\right)^2\right]^{\frac{1}{2}}.$$

Also

$$\left.\begin{aligned} x &= x_0 + \frac{\alpha}{\omega}\sin(\omega t + \epsilon) \\ y &= y_0 - \frac{cX}{eH}t - \frac{\alpha}{\omega}(1 - \cos(\omega t + \epsilon)) \\ z &= z_0 + \beta t \end{aligned}\right\} \quad . \quad . \quad (14)$$

Hence, if we consider ions for definiteness, so that $\omega > 0$, the top of the motion corresponds to $\sin(\omega t + \epsilon) = 1$ and the lowest point to

$\sin(\omega t + \epsilon) = -1$. Thus the velocity at the top of the motion parallel to O$y$ is

$$\dot{y} = -\frac{c\mathrm{X}}{e\mathrm{H}} - \alpha,$$

and at the bottom is

$$\dot{y} = -\frac{c\mathrm{X}}{e\mathrm{H}} + \alpha.$$

As $\frac{c\mathrm{X}}{e\mathrm{H}}$ is negative and is usually small compared with $\alpha$, the motion at the top of the trajectory is in general opposed to the drift. Hence we can see that equilibrium is secured thus. At any point there are a number of particles near the top of their motions and a number near the bottom of their motions. The first set has a mean velocity opposed to the drift, the second a slightly larger mean velocity parallel to the drift. Owing to the pressure gradient the first set is slightly more numerous than the second. Thus the net momentum of the two sets can vanish.

9. We shall establish rigorously the result thus loosely stated. First we note that, as the state is stationary, the number of particles which begin free paths in a given region $dx\,dy\,dz$ with velocities in the range $du\,dv\,dw$ in a small time $dt$ is equal to the number which in the same time end their free paths in the same region with velocities in the same range. These may be regarded as continuing the paths of the others; hence collisions can be neglected in any macroscopic consideration.

Let us neglect collisions, therefore. Let us suppose that the number of particles whose quantities $\alpha$, $\beta$, $\epsilon$, $x_0$, $y_0$, $z_0$ lie in the ranges $d\alpha$, $d\beta$, $d\epsilon$, $dx_0$, $dy_0$, $dz_0$ at any given time is

$$\mathrm{F}(\alpha, \beta, x_0)d\alpha\,d\beta\,d\epsilon\,dx_0\,dy_0\,dz_0.$$

Then in a phase element $dx\,dy\,dz\,du\,dv\,dw$ there are

$$\frac{1}{\alpha}\mathrm{F}(\alpha, \beta, x_0)dx\,dy\,dz\,du\,dv\,dw,$$

where $x_0$, $\alpha$, $\beta$ are connected with $x, y, z, u, v, w$ by equations (13) and (14).

Thus the momentum per unit volume parallel to O$y$ is

$$m\iiint\frac{1}{\alpha}\mathrm{F}(\alpha, \beta, x_0)v\,du\,dv\,dw$$

$$= m\iiint\mathrm{F}\Big(\alpha, \beta, x - \frac{\alpha}{\omega}\sin(\omega t + \epsilon)\Big) \times\Big[-\frac{c\mathrm{X}}{e\mathrm{H}} - \alpha\sin(\omega t + \epsilon)\Big]d\alpha\,d\beta\,d\epsilon.$$

We have to the first order in $\frac{\alpha}{\omega}$

$$\mathrm{F}\Big(\alpha, \beta, x - \frac{\alpha}{\omega}\sin(\omega t + \epsilon)\Big) = \mathrm{F}(\alpha, \beta, x) - \frac{\partial\mathrm{F}(\alpha, \beta, x)}{\partial x}\frac{\alpha}{\omega}\sin(\omega t + \epsilon),$$

on making the characteristic assumption of diffusion problems. Then, as

$$\int_0^{2\pi} \sin(\omega t + \epsilon) d\epsilon = 0$$

$$\int_0^{2\pi} \sin^2(\omega t + \epsilon) d\epsilon = \pi,$$

the expression for the momentum becomes

$$-m\iiint\left[\mathrm{F}(\alpha, \beta, x)\frac{c\mathrm{X}}{e\mathrm{H}} - \frac{1}{2}\frac{a^2}{\omega}\frac{\partial \mathrm{F}(\alpha, \beta, x)}{\partial x}\right] d\alpha\, d\beta\, d\epsilon.$$

Now we have for the number of these particles per unit volume

$$\begin{aligned} n &= \iiint \frac{1}{a}\mathrm{F}(\alpha, \beta, x_0) du\, dv\, dw \\ &= \iiint \mathrm{F}\left(\alpha, \beta, x - \frac{a}{\omega}\sin(\omega t + \epsilon)\right) d\alpha\, d\beta\, d\epsilon \\ &= \iiint \left[\mathrm{F}(\alpha, \beta, x) - \frac{a}{\omega}\sin(\omega t + \epsilon)\frac{\partial \mathrm{F}(\alpha, \beta, x)}{\partial x}\right] d\alpha\, d\beta\, d\epsilon \\ &= \iiint \mathrm{F}(\alpha, \beta, x) d\alpha\, d\beta\, d\epsilon. \end{aligned}$$

Also the pressure is

$$\begin{aligned} p &= \iiint \frac{mu^2}{a}\mathrm{F}(\alpha, \beta, x_0) du\, dv\, dw \\ &= \iiint ma^2 \cos^2(\omega t + \epsilon)\mathrm{F}\left(\alpha, \beta, x - \frac{a}{\omega}\sin(\omega t + \epsilon)\right) d\alpha\, d\beta\, d\epsilon \\ &= \iiint ma^2 \cos^2(\omega t + \epsilon)\left\{\mathrm{F}(\alpha, \beta, x) - \frac{a}{\omega}\sin(\omega t + \epsilon)\frac{\partial \mathrm{F}(\alpha, \beta, x)}{\partial x}\right\} d\alpha\, d\beta\, d\epsilon \\ &= \tfrac{1}{2}\iiint ma^2 \mathrm{F}(\alpha, \beta, x) d\alpha\, d\beta\, d\epsilon. \end{aligned}$$

Hence the momentum is

$$-mn\frac{c\mathrm{X}}{e\mathrm{H}} + \frac{1}{\omega}\cdot\frac{\partial p}{\partial x} \quad . \quad . \quad . \quad . \quad (15)$$

The first term is the amount which Chapman predicted. The second is the correction which results when we consider a pressure gradient. It will be noted that we have made no use of Maxwell distributions in this deduction.

If the mean velocity of a particle is $v_0$, then

$$mv_0 n = -mn\frac{c\mathrm{X}}{e\mathrm{H}} + \frac{mc}{e\mathrm{H}}\frac{\partial p}{\partial x}.$$

In particular, if $\frac{\partial p}{\partial x} = +nX$ we have $v_0 = 0$. Thus in this case the drift currents are completely masked by the effect of the pressure gradient.

10. We will now return to a consideration of the results (12). We observe that the equation of dissociative equilibrium can now be satisfied without any assumption as to the value of E. For from equations (12) we deduce the equations

$$\left.\begin{aligned} \frac{dp}{dx} &= -n(m_i + m_e)g \\ \frac{dp_i}{dx} &= -n_i\left(m_i g - e\mathrm{E} - \frac{e\mathrm{H}v_0}{c}\right) \\ \frac{dp_e}{dx} &= -n_e\left(m_e g + e\mathrm{E} + \frac{e\mathrm{H}v_0}{c}\right) \end{aligned}\right\} \quad . \quad . \quad (16)$$

whence the condition of dissociative equilibrium (9) is satisfied identically.

These equations also give us some idea as to the magnitude of the alteration in the Pannekoek-Rosseland electric field. For if we assume $n_i/n_e$ is constant we get from equations (16)

$$m_i g - e\left(\mathrm{E} + \frac{\mathrm{H}v_0}{c}\right) = m_e g + e\left(\mathrm{E} + \frac{\mathrm{H}v_0}{c}\right) \quad . \quad . \quad (17)$$

Thus E is changed into $\mathrm{E} + \frac{\mathrm{H}v_0}{c}$. Taking $v_0$ as the velocity at the Sun's equator due to rotation, we have $v_0 = 2 \times 10^5$ cms. sec.$^{-1}$. Thus the alteration is of the order of $(6{\cdot}6 \times 10^{-6} \times \mathrm{H})$ E.S.U., H being measured in gauss. This is large compared with E, which is of the order of $6{\cdot}5 \times 10^{-7}$ volts/cm.* $= 2{\cdot}2 \times 10^{-9}$ E.S.U.

11. It may be urged that the work of § 7 is rendered invalid by the bounding surface introduced, which has no physical justification. A model resembling the facts a trifle more closely is as follows. Consider a mass of gas enclosed in a surface of revolution whose axis is $Oz$. We suppose this bounding surface to be smooth and such that impacts with it are elastic. Let us seek a steady motion in which the gas revolves with constant angular velocity $\omega$ under its own gravitational, electrostatic, and magnetic forces. These will be axial, so that

$$\mathrm{H}_x/x = \mathrm{H}_y/y.$$

Also as H is the quantity commonly called the magnetic induction,

$$\mathrm{div}\ \mathrm{H} = 0.$$

Hence a function $\xi$ of $z$ and $\sqrt{(x^2 + y^2)}$ can exist, such that

$$\frac{\partial \xi}{\partial x} = -x\mathrm{H}_z, \qquad \frac{\partial \xi}{\partial y} = -y\mathrm{H}_z, \qquad \frac{\partial \xi}{\partial z} = x\mathrm{H}_x + y\mathrm{H}_y \quad . \quad (18)$$

* Milne, *Proc. Camb. Phil. Soc.*, **22**, 504 (1925).

The equations of motion of an ion are now

$$m_\iota \ddot{x} = -m_\iota \frac{\partial \phi}{\partial x} - e\frac{\partial \psi}{\partial x} + \frac{e}{c}(\dot{y}\mathrm{H}_z - \dot{z}\mathrm{H}_y), \text{ etc.}$$

where $\phi$, $\psi$ are the gravitational and electrostatic potentials; and we have similar equations for an electron. Using these, we find the velocity distributions

$$\left.\begin{aligned} &\mathrm{C}\exp\{-\tfrac{1}{2}hm(u^2+v^2+w^2) + hm\omega(uy-vx) - hm\phi\} \\ &\mathrm{C}_\iota \exp\left\{-\tfrac{1}{2}hm_\iota(u^2+v^2+w^2) + hm_\iota\omega(uy-vx) - hm_\iota\phi - he\psi + \frac{he\omega}{c}\xi\right\} \\ &\mathrm{C}_e \exp\left\{-\tfrac{1}{2}hm_e(u^2+v^2+w^2) + hm_e\omega(uy-vx) - hm_e\phi + he\psi - \frac{he\omega}{c}\xi\right\} \end{aligned}\right\} \quad (19)$$

These give equations of the form

$$\begin{aligned} n &= \mathrm{B}\exp\{\tfrac{1}{2}hm(x^2+y^2) - hm\phi\} \\ n_\iota &= \mathrm{B}_\iota \exp\left\{\tfrac{1}{2}hm_\iota(x^2+y^2) - hm_\iota\phi - he\psi + \frac{he\omega}{c}\xi\right\} \\ n_e &= \mathrm{B}_e \exp\left\{\tfrac{1}{2}hm_e(x^2+y^2) - hm_e\phi + he\psi - \frac{he\omega}{c}\xi\right\}. \end{aligned}$$

By a suitable choice of B, $\mathrm{B}_\iota$, $\mathrm{B}_e$ the equation of dissociative equilibrium can be satisfied identically.

We can see from these values of $n$, $n_\iota$, and $n_e$ why it was necessary that we should have a boundary surface. Otherwise, as $x^2+y^2 \to \infty$, $n$, $n_\iota$, $n_e$ would not remain finite. In the case of the Sun this does not occur, as the Sun does not spin uniformly. We can suppose that the layers at a sufficient distance from the Sun lag behind those closer in. But this certainly occurs outside the region in which the effective limitation of the magnetic field takes place: hence, if we take the bounding surface outside this region we have a fair approximation to the actual conditions. It is also fairly clear that, as on the boundary the values of $n$, $n_\iota$, and $n_e$ will be small compared with their values in the region of limitation, the effect of the boundary can hardly be held to be an adequate explanation of the difference between Chapman's theory and ours.

We cannot fail to notice the difference between the functions of the boundary in the case treated in § 1 and in that treated here. In that, the boundary surface played a large part in the maintenance of the Maxwell distribution: in this, there is no singularity at the centre to correspond. It is probable that this is due to the fact that Ross Gunn's second effect reduces Chapman's effect to zero as we approach the centre.

We may also remark that in both the present case and that of § 7, Ross Gunn's second effect, as well as Chapman's, has failed to appear. It is probable that this is due to a similar fact to that which caused Chapman's effect to disappear—that is, that the mean motion

at the top of the path is again opposed to that at the bottom, and that the method of averaging was again at fault, as it was grouping together a velocity in one direction at one level and a velocity in the opposite direction at a different level.

12. It must be emphasised that we have been considering only steady states. As we depart from these, so the theory is liable to fail. For example, the theory needs to be examined with regard to the temperature gradient of the Sun, the vortical motion near sunspots, and the fluctuating conditions in the Earth's atmosphere, before we can apply it without restriction to the Sun, sunspots, and the Earth's magnetic field. As Ross Gunn's work on the second and third of these depends on a non-steady state, it may still remain valid.

*Summary.*

1. The theories of Ross Gunn and Chapman as to the limitation of the Sun's magnetic field are set out and discussed.

2. It is shown that the theory of Ross Gunn breaks down owing to the lack of a boundary effect.

3. It is shown that Chapman's equations, when corrected for the mechanical effect of the magnetic forces, lead to a contradiction with the second law of thermodynamics, unless H = 0.

4. It is shown that in a steady state neither Chapman's nor Ross Gunn's effects appear. This is shown to be due to a masking caused by the pressure gradient.

5. New mechanical equations are set up and the modification of the Pannekoek-Rosseland field due to the magnetic field is found.

6. The theory is one of steady states; it may break down if we consider the temperature gradient of the Sun.

In conclusion, I should like to express my thanks to Professor Milne, who suggested the problem, and without whose valuable assistance and criticisms this paper could not have been written.

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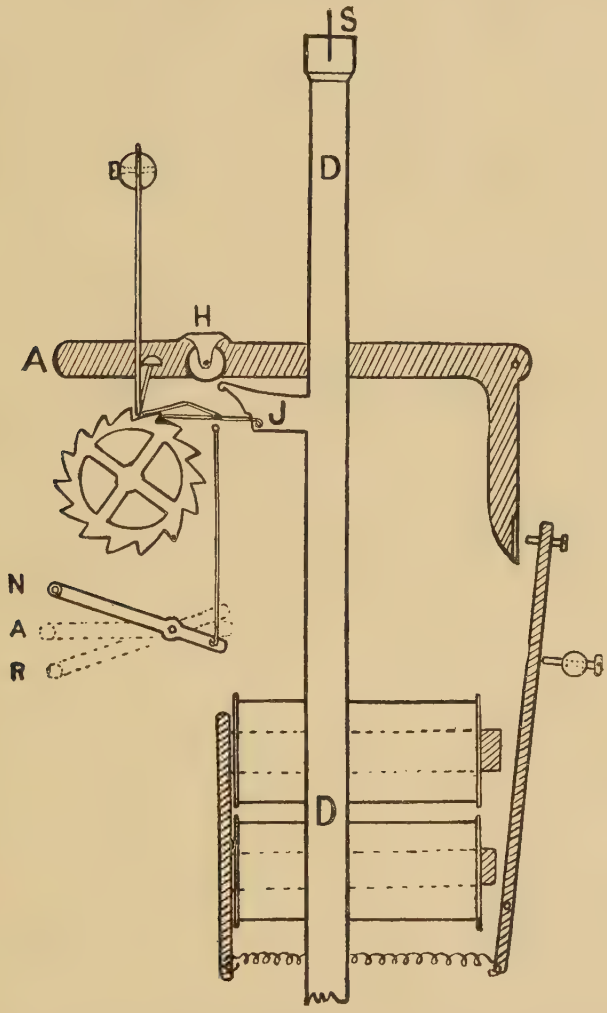
*Registering the Oscillations of a Pendulum without touching it.*
By F. Hope-Jones.

In his note in *M.N.*, **89**, 713, 1929, upon registering the oscillations of a pendulum without disturbing it, General Ferrié states the case against clicks, ratchet wheels and their contacts clearly and forcibly, but he also states that the problem has not been solved, and proceeds to discuss photo-electric cells and triode valves as substitutes.

It is with some timidity that I venture to differ from this distinguished scientist, to whom the world is so much indebted for the rhythmic time signals; but the prestige of British clockmakers is involved, and it is a duty, coupled perhaps with a little natural pride, to point out that this problem was solved long ago, as will be seen

from the diagram, reproduced from the *British Horological Journal* of 1906 January. It will be observed that the roller H on the gravity arm A falls off the end of the pallet J after imparting its




impulse to the pendulum, and alights upon the armature, thus making contact, consequently the moment of contact corresponds with the identical instant of the period of the pendulum, and there is no interference with the latter. The precise point of time at which the contact occurs is dependent solely upon the pendulum and upon gravity, and it is therefore identical at each operation.

It is only recently that scientists have become aware of the completely satisfactory nature of this solution, their attention having been drawn to it by the Loomis spark chronograph, described in the *Monthly Notices* for 1929 April.

General Ferrié may therefore be pardoned for having overlooked it; but it is difficult to exonerate *la profession d'horlogerie française* for their ignorance of this and of the other equally vital principles involved in this construction, which has contributed so substantially to the science of accurate time measurement in this country. It is indeed disappointing to hear that French clockmakers are still attempting to make contacts from pendulum-propelled ratchets and are still forcing the geophysicists and the astronomers to seek for feeble substitutes.

It is a characteristic of *l'horlogerie française* that in their applications of electricity to horology they invariably interfere with their pendulums or wheelwork for contact-making purposes. I abjured the evil habit of making contacts from wheelwork thirty-five years ago, and I look upon photo-electric cells and triode valves as but ineffectual substitutes for substantial switches. I set out in 1895 in the diametrically opposite direction, recognising that switching was a brute-force operation, and that it must be effected by the pendulum itself without interfering with it in the slightest degree. This particular feature is one of many which are vital to the successes recently achieved in time measurement at Greenwich and elsewhere; it is the means by which the Shortt free pendulum transmits its synchronising signal to its slave clock, as well as the source of its remontoire, and nothing less perfect would suffice. It is also the means by which the Rugby time signals are transmitted, and its time-spacing is correct to within $\pm$ $0^{s}\cdot001$.

A curious and valuable feature of this switch is that it is electrically interlocked with the apparatus it is required to operate. The latter is bound to receive sufficient current for sufficient time to enable it to perform its duty; it is impossible for the prime mover to operate at all without it giving that assurance, whereas all such methods as that described by the distinguished General are open to the same objection as the ordinary clock contacts which he condemns, in that they are arbitrary in their action and are dependent upon predetermined factors which may or may not be adequate for the work in hand.

The Loomis spark chronograph divides each second into one thousand separate compartments or cells, and the switch I have described places its spark into the same one every time.

I can conceive it possible that still smaller divisions of time might be required for such purposes as the measurement of radio frequencies, in which case it may be that the ingenious method of General Ferrié might justify itself, although he apparently only claims the same degree of accuracy commonly achieved by the Synchronome switch.

## *The Meteor Swarm of Halley's Comet.*

By R. A. McIntosh.

The meteors which radiate in April and May from the constellation of Aquarius have rarely been observed satisfactorily or in large numbers. This may be explained by the fact that almost all meteor observers in the past have been situated in the Northern Hemisphere, and the Aquarid radiant is thus rather unfavourably placed for observation by them. The fact that only eight radiants for this rich stream are given in Denning's *General Catalogue of Radiant Points*, and that not a great deal has been added to our knowledge since that excellent work was published, is sufficiently illustrative of this. In view of our scanty knowledge as far as this particular shower is concerned, the results obtained by the writer at Auckland, New Zealand, in 1929 are of interest to meteor observers, being probably the most extensive series of observations of this shower yet secured in a single year.

An observer working in the southern hemisphere naturally has a distinct advantage over his northern contemporaries in that the radiant is more favourably placed for observation and is visible for a longer period each night. These advantages are somewhat discounted in New Zealand by the shower occurring during a very rainy month. On this account, attempts to observe the shower prior to 1929 have not been very successful, but in May of this year the observing conditions, while far from perfect, permitted a very satisfactory investigation of the character of the Aquarid shower.

The following table gives details of watches carried out at Auckland during the activity of the Aquarid shower :—

| Date. 1929. | | G.M.T. Began. h | m | Ended. h | m | Total. m | No. of Meteors. | Rate. | Factor. | Corr. Rate. | Aqrd. Rate. | Remarks. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| May | 2 | 15 | 30 | 18 | 00 | 150 | 34 | 14 | 0·5 | 28 | 12 | Moon 22d., near radiant ; clear. |
| ,, | 3 | 15 | 10 | 16 | 30 | 55 | 8 | .. | .. | .. | .. | Moon 23d., in Aqr. ; passing clouds. |
| ,, | 5 | 15 | 00 | 18 | 00 | 180 | 50 | 17 | 0·8 | 21 | 8 | Moon 25d. Misty, some clouds. |
| ,, | 6 | 15 | 00 | 16 | 10 | 70 | 13 | 11 | 0·6 | 18 | 10 | Misty ; half sky clouded. |
| ,, | 7 | 15 | 00 | 18 | 00 | 155 | 57 | 22 | 1·0 | 22 | 10 | Clear ; misty at end. |
| ,, | 8 | 15 | 07 | 18 | 10 | 148 | 46 | 18 | 0·9 | 20 | 9 | Clear ; then cloudy. |
| ,, | 9 | 15 | 20 | 15 | 35 | 15 | 6 | .. | .. | .. | .. | Heavy clouds. |
| ,, | 10 | 14 | 50 | 16 | 10 | 60 | 12 | 12 | 0·6 | 20 | .. | Passing clouds ; misty. |
| ,, | 11 | 12 | 55 | 18 | 10 | 315 | 80 | 15 | 1·0 | 15 | 8 | Clear. |
| ,, | 12 | 14 | 55 | 15 | 20 | 25 | 5 | .. | .. | .. | .. | Almost totally clouded. |
| | | | | | | 19h 33m | 311 | | | | | |

The observed rate per hour (column 6) has been corrected by a factor (column 7), making some allowance for such hindrances to observing as clouds and moonlight. This factor has always been estimated on a very conservative basis. The horary rate of all meteors under perfect conditions is then found in column 8. It is interesting to note the good agreement between these final corrected rates, all of which were arrived at quite independently of each other. This is an indication that the use of a corrected rate was justified. The horary rate of the Aquarid shower, similarly corrected, is given in column 9. This shows that the maximum occurred early in May, when bad weather prevented any observations being made. The rate for all meteors, after May 2, seemed to remain constant at about 20 an hour.

The first meteor seen which appears to have been definitely connected with the shower was observed on April 21. The last meteors from the shower were seen on May 12, when the stream was still prominently visible, despite the almost totally clouded sky. It is reasonable to assume that the shower continued in fair activity for two or three days after this, but further observations were, unfortunately, prevented by bad weather.

We find, therefore, that meteors were encountered over the period April 21 to May 12, and perhaps even later, during which time the Earth moved through 21° in its orbit, or roughly 35,000,000 miles, passing diagonally across the meteor stream. The great thickness of the swarm is a good illustration of the dispersive action of the planets during the centuries that the parent comet has been in existence.

The practice followed by the writer in observing meteors is to plot each meteor's path when observed on special gnomonic maps, at the same time entering on record-sheets such details of physical appearance, etc., as are required for statistical studies. When examining the observations and deducing radiants, each night's results are treated separately, and radiants secured on different nights are not compared until all the observations have been worked up and all radiants independently determined. There is, therefore, very little possibility of bias entering into the deductions.

The following table gives particulars of the various Aquarid radiants observed :—

| No. | G.M.T. | | R.A. | Dec. | Meteors. | Weight. | Remarks. |
|---|---|---|---|---|---|---|---|
| | | | ° | ° | | | |
| 1 | 1929 May | 2·70 | 334·0 | −1·5 | 18 | Good | Diffuse. |
| 2 | ,, ,, | 5·69 | 337·5 | −1·0 | 19 | ,, | ,, |
| 3 | ,, ,, | 6·65 | 338·6 | 0·0 | 6 | Fair | Uncertain in declination. |
| 4 | ,, ,, | 7·69 | 339·0 | −0·4 | 24 | Good | Diffuse. |
| 5 | ,, ,, | 8·69 | 339·0 | +0·3 | 10 | ,, | ,, |
| 6 | ,, ,, | 11·69 | 342·7 | +2·5 | 21 | ,, | ,, |

Owing to the poor observing conditions the following numbers of

Aquarids were seen on the other dates, insufficient in each case for the determination of reliable radiants :—

| Date. | Aquarids. | Date. | Aquarids. |
|---|---|---|---|
| April 21 | 1 | May 10 | 1 |
| May 3 | 4 | „ 12 | 2 |
| „ 9 | 1 | | |

On all occasions the radiants obtained were rather diffuse in nature, the radiant area generally covering about 5° in each co-ordinate. It cannot be doubted that this diffuseness is real, and is not a result of faulty observation of individual paths. In spite of the wide radiant areas it was not difficult to deduce centres of radiation for the showers, as on each date the majority of meteors emanated from a fairly well-defined point, around which the non-conforming meteors were uniformly scattered.

Perhaps the most interesting feature revealed in the above table is the unmistakable shift of the centre of radiation during the period covered by the observations. It should not be overlooked, also, that the Aquarid meteor seen on April 21 confirms the shift shown in the table and doubles it in amount.

Though one would naturally expect the radiant-point of a meteor shower definitely connected with a comet to be a moving one, previous observations have not been able to demonstrate this shift very satisfactorily. Indeed, the only previous evidence of motion known to the writer is in the work of Dr. Chas. P. Olivier,* who in various years since 1910 has secured observations showing an eastward movement of radiants. While certain selected radiants whose orbits agreed fairly well with that of Halley's comet showed a definite shift in Right Ascension, there were other radiants, apparently equally well determined, which gave no support to this movement. Four radiants secured by Dr. Olivier and R. M. Dole in 1921 † considerably strengthened his claim that the radiant was in motion, and the additional support given by the radiants observed by the writer in 1929 must place this movement beyond dispute.

In view of the connection of this meteor shower with Halley's comet, elliptical orbits were calculated for each radiant, on the assumption that the meteors possessed a semi-major axis identical with that of the comet. The zenith attraction of the radiants never amounted to as much as 1°, and for all practical purposes could have been neglected, but in computing the orbits it was deemed advisable to make a correction for this attraction, the value of which was found by a short graphical process sufficiently accurate for the purpose. The orbits, however, were rigorously computed with four-figure logarithms, the final results being rounded off to tenths of a degree.

The following elliptical orbits were obtained, and the orbit of

* " 175 Parabolic Orbits . . .," *Trans. Amer. Phil. Soc.*, N.S. **22**, part i, 1911.

† *Observatory*, **44**, 242, 1921.

Halley's comet, and the elements of the meteor stream calculated by Hoffmeister,* are included for purposes of comparison.

| | *i.* | *q.* | $\pi$. | $\Omega$. | $\omega$. | *e.* | Log *a.* | Wt. |
|---|---|---|---|---|---|---|---|---|
| | ° | | ° | ° | ° | | | |
| Comet 1910 II. | 162·2 | 0·587 | 169·0 | 57·3 | 111·7 | 0·968 | 1·254 | |
| Hoffmeister | 162·6 | 0·595 | 144·9 | 45·2 | 99·7 | 0·967 | 1·254 | |
| 1929, May 2·70 | 161·6 | 0·591 | 140·8 | 41·8 | 99·0 | 0·967 | 1·254 | 3 |
| ,, ,, 5·69 | 162·9 | 0·559 | 139·8 | 44·7 | 95·1 | 0·969 | 1·254 | 3 |
| ,, ,, 6·65 | 161·6 | 0·526 | 137·3 | 45·6 | 91·6 | 0·971 | 1·254 | 1 |
| ,, ,, 7·69 | 163·0 | 0·566 | 142·6 | 46·6 | 96·0 | 0·968 | 1·254 | 3 |
| ,, ,, 8·69 | 161·9 | 0·590 | 143·5 | 47·6 | 95·9 | 0·967 | 1·254 | 2 |
| ,, ,, 11·69 | 159·1 | 0·483 | 143·7 | 50·5 | 93·2 | 0·973 | 1·254 | 3 |
| Mean | 161·7 | 0·553 | 141·3 | 46·1 | 95·1 | 0·969 | 1·254 | |

The excellent agreement between the various orbits enumerated above demonstrates the authenticity of the shift in the radiant. The radiant for May 11 was apparently too far north, which resulted in its agreement not being as good as in other cases. If I had cared to select a position for the radiant on the southern edge of the diffuse area the agreement would have been much improved, but it was deemed advisable to give the position of the centre, as had been done in the other cases.

The individual meteors of the Aquarid shower presented widely differing characteristics. All colours were represented, with a preference for meteors ranging from green to white. The magnitudes on the whole seemed brighter than the average meteor, but this may have been in part due to the adverse observing conditions and the moonlight during the earlier part of the month. Eight meteors out of the 103 Aquarids seen were brighter than the first magnitude. A large number were trained, and their apparent velocities were rather swift.

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*Observations of the Orionid Meteors.* By R. A. McIntosh.

The 1928 Orionid meteoric display occurred under favourable conditions for observation, the Moon not being troublesome until the last week of the month. The weather locally was far from perfect at that time, but was much better than has been the case in Auckland during past years.

* *Astr. Nach.*, **196**, 309, 1913.

The following watches were made :—

TABLE I.

*Details of Watches.*

| Date. | G.M.T. Began. h m | G.M.T. Ended. h m | Total Watch. m | No. of Meteors. | Orionids. | Sky. |
|---|---|---|---|---|---|---|
| 1928 Oct. 14 | 14 45 | 15 35 | 50 | 7 | 0 | Clouds at times. |
| 15 | 14 36 | 16 26 | 110 | 41 | 16 | Clear. |
| 16 | 14 25 | 16 21 | 116 | 37 | 10 | Misty at times. |
| 18 | 14 50 | 16 25 | 95 | 16 | 4 | Clouded.* |
| 20 | 13 42 | 15 50 | 128 | 34 | 12 | Clear. |
| 23 | 15 0 | 15 54 | 54 | 10 | 3 | Heavy mist. |
| 24 | 14 35 | 16 15 | 100 | 19 | 9 | Clear; Moon. |
| | | | $10^h\ 53^m$ | 164 | 54 | |

*Method of Observation.*—In view of the controversy which centres upon the Orionid radiant I carefully planned my scheme of observing in advance, considering the special features to be looked for. On ordinary occasions a meteor observer must pay considerable attention to a large area of sky, endeavouring to find all the radiants in activity in the area at that time, and care must also be taken over a number of minor details (colours, durations, etc.). In the case of the Orionids, where the principal need was accurate radiants, I decided to pay little attention to the meteors from minor showers, to observe only in that portion of the sky where the Orionids would be most prolific, and to devote less time than usual to accurate determinations of the minor details. This left me free to concentrate on the directions of flight of the Orionids.

Unlike my usual work in these skies where the radiants are not known before observing, I knew that the radiant should lie near the faint stars $\nu$ and $\xi$ Orionis. Thus, when each meteor flashed into view, I was able to place the string I was using along its apparent path to determine the beginning and end points, and, in addition, glancing quickly back toward the stars near the radiant, to determine their relation to each meteor's line of flight. The projecting of the paths on the sky itself would tend to make the plotting of the paths on the maps more accurate.

As soon as observed, each path was drawn on one of a set of gnomonic maps of the region. Fresh maps were used each night, and the earlier maps were not looked at until October 20, after I had obtained my three best radiants. The remaining maps were not examined until the final observations had been made. The excellent progression in right ascension shown by the successive radiants could not, therefore, have been due to knowledge of the previously determined positions.

*Physical Appearance of Meteors.*—The Orionids, as observed in

* All the meteors seen on this date were observed through cloud gaps. The sky was clear only for five minutes.

Europe and America, have been described as swiftly moving meteors leaving phosphorescent streaks. A New Zealand observer, guided by Mr. Denning's words in *Telescopic Astronomy,* would expect long-pathed meteors with slow apparent velocity, owing to the considerable zenith distance of the radiant (never less than 52°, and more than that at the times I observed). The existence of this type of meteor from the shower has been shown in my observations of 1927 (*Popular Astronomy,* **36**, 64) and 1928.

The Orionids in 1928 proved the most easily recorded meteors I have yet observed. The majority were long-pathed, bright, slow-travelling objects, leaving behind luminous trains which persisted sufficiently long to enable me to place the string along their apparent paths before the trains had vanished. This resulted in splendid determinations of paths and consequently well-defined radiants. The following tables will serve to illustrate the physical appearance of the meteors :—

TABLE II.

*Physical Details of Orionids.*

| Date, | Meteors. | Orionids. | Average Duration. | Number Trains. | Average Duration Trains. | Colour. | | Rate. |
|---|---|---|---|---|---|---|---|---|
| | | | s | | s | Red. | Blue. | |
| 1928 Oct. 14 | 7 | 0 | .. | .. | .. | .. | .. | 0 |
| 15 | 41 | 16 | 0·6 | 11 | 1·2 | 4 | 0 | 8 |
| 16 | 37 | 10 | 0·5 | 4 | 1·6 | 3 | 1 | 5 |
| 18 | 16 | 4 | 0·6 | 0 | .. | 1 | 0 | 3 |
| 20 | 34 | 12 | 0·7 | 8 | 1·1 | 5 | 3 | 6 |
| 23 | 10 | 3 | 0·6 | 1 | 1·4 | 0 | 0 | 3 |
| 24 | 19 | 9 | 0·4 | 1 | 1·0 | 0 | 1 | 5 |
| Totals | 164 | 54 | 0·6 | 25 | 1·3 | 13 | 5 | |

The final column is an approximation. It was impossible to arrive at the date of maximum activity owing to the broken nature of my observations.

TABLE III.

*Magnitudes of Orionids.*

| Date. | Magnitude. | | | | | |
|---|---|---|---|---|---|---|
| | <1. | 1. | 2. | 3. | 4. | 5. |
| 1928 Oct. 15 | 3 | 2 | 3 | 6 | 1 | 1 |
| 16 | 2 | 2 | 2 | 2 | 2 | 0 |
| 18 | 0 | 0 | 0 | 3 | 0 | 1 |
| 20 | 2 | 2 | 3 | 4 | 0 | 1 |
| 23 | 0 | 1 | 0 | 1 | 1 | 0 |
| 24 | 0 | 0 | 1 | 4 | 2 | 2 |
| Totals | 7 | 7 | 9 | 20 | 6 | 5 |
| Percentages | 13 | 13 | 17 | 37 | 11 | 9 |

*Orionid Radiants.*—The various Orionid radiants found from my maps are given in Table IV. These positions are not corrected for zenith attraction, which amounted to only $\frac{1}{2}^\circ$, and was therefore within the errors of observation.

TABLE IV.

*Radiants of Orionids.*

| No. | Date. | Radiant. | No. of Meteors. | Remarks. | |
|---|---|---|---|---|---|
| 1 | 1928 Oct. 15·65 | 89°·2, + 14°·2 | 15 | Good. | |
| 2 | 16·64 | 90°·0, + 15°·0 | 9 | Good. | |
| 3 | 18·65 | 91°·6, + 14°·6 | 4 | Poor. | |
| 4 | 20·62 | 92°·5, + 14°·2 | 12 | Good. | |
| 5 | 23·64 | 93°·5, + 14°·9 | 3 | ? | |
| 6–7 | 24·64 | 99°·4, + 13°·7 | 9 | Good. | Mean of 2 radiants. |

The above radiant-points were mostly well-defined centres, and only meteors observed on one night were used to determine each. No meteor which passed more than 1° from the adopted centre was used.

The custom of some observers of using a number of radiants surrounding the "true" radiant, and each consisting of very few meteors, has received strong criticism, the view being advanced that one radiant-point more accurately interprets the observations. I have been careful to follow the latter course in determining my radiants, although it is my strong personal opinion that there were two centres of activity in the same R.A. and 1° apart in dec. (especially on October 15 and 24) which moved together in R.A. from day to day. However, to prevent criticism on this score, I have given above only the centres of radiation. My observations disclose that the radiants were without doubt clearly diffuse in declination, as was shown by a number of excellently determined meteors moving in R.A. on the earlier dates, and on October 24 by a stationary meteor and one other giving a radiant at 100°·1, + 14°·2. The other seven Orionids seen on that date radiated from 99°·1, + 13°·5.

The radiant numbered 4 in Table IV. has undoubtedly been placed slightly too close to the star $\nu$ Orionis, a tendency known to all meteor observers, and accentuated by the method of observing adopted. The radiant on October 23 is worthless, its R.A. depending on only one meteor.

Table V. gives the orbits of the various radiants, which in this case have been corrected for zenith attraction. Parabolic motion has been assumed for the meteors.

There is an excellent agreement between the various orbits, making it almost certain that they refer to the one stream of meteors. Errors in the latitudes of my radiants, which I have already stated to be certainly diffuse in declination, would account in some degree for the variations shown in the longitude of perihelion. The longitude of the

TABLE V.

*Orbits of the Radiants.*

| Rad. No. | L. | *i*. | *q*. | $\pi$. | $\Omega$. |
|---|---|---|---|---|---|
| | ° | ° | | ° | ° |
| 1 | 113·0 | 162·0 | 0·613 | 98·8 | 22·0 |
| 2 | 113·9 | 163·5 | 0·602 | 100·9 | 23·0 |
| 3 | 115·9 | 162·6 | 0·595 | 103·7 | 25·0 |
| 4 | 117·9 | 161·6 | 0·568 | 109·0 | 27·0 |
| 6–7 | 121·9 | 161·9 | 0·642 | 104·1 | 31·0 |
| Mean Orbit | | 162·3 | 0·604 | 103·3 | 25·6 |

ascending node depends on the position of the Earth, and therefore varies from day to day. L is the longitude of the apex of the Earth's motion.

All the above radiants for which orbits have been computed agree closely with the radiants observed by R. M. Dole (*Popular Astronomy*, **31**, 37, 1923) and Dr. C. P. Olivier (*Observatory*, **46**, 17, 1923). The movement of the radiant has also been shown in the various publications of the American Meteor Society,* and is confirmed in an indisputable manner by a photographic radiant secured at Harvard Observatory in 1922 (*H.C.O. Bulletin*, 778).

*Possibility of Stationary Radiation.*—Advocates of stationary radiation believe that during October two stationary radiants are in activity—the Orionids at 91°, + 15° and the Geminids only 6° east of this at 97°, + 15°. I have carefully examined all my maps to ascertain if they give any support to this belief. The following table, for which I used only those meteors capable of indicating the R.A. of the radiant, explains my results. Any meteor which in any part of its path passed within a circle 2° in diameter surrounding either of the two points has been accepted as coming from that point.

TABLE VI.

*Existence of Two Stationary Radiants.*

| Date. | Meteors from 91°, + 15°. | Meteors from 97°, + 15°. |
|---|---|---|
| 1928 Oct. 15 | 4 | 0 |
| 16 | 5 | 0 |
| 18 | 2 | 0 |
| 20 | 5 | 0 |
| 23 | 0 | 0 |
| 24 | 0 | 1 |

* "175 Parabolic Radiants . . .," *Transactions American Philosophical Society*, **22**, part 1, 1911; "126 Parabolic Orbits . . .," *Publications Leander McCormick Observatory*, **2**, part 4, 1914; "349 Parabolic Orbits . . .," *Publications Leander McCormick Observatory*, **2**, part 7, 1919.

On first glance Table VI. would appear to support at least a stationary radiant at 91°, + 15°, but this is merely due to a number of meteors which were so placed that their paths passed through the circle surrounding the radiant, although there were no intersecting paths within the circle itself except on October 18, when the actual radiant was near the adopted position. On seven nights only one meteor could be found passing near the supposed Geminid radiant, and there was not the slightest evidence for the late activity of the 91°, + 15° centre.

*Minor Showers.*—There were a number of minor showers active at the time of the Orionid display which may possibly have some connection with the principal stream. Similar minor showers have been noticed by the writer during displays of Aquarids, Perseids, Orionids, and Leonids, and may be a result of the dispersive action of the earth on the various meteor streams.

Table VII. gives the positions of such minor showers as were observed in the course of watching for Orionids.

TABLE VII.

*Minor Radiants.*

| No. | Date. | Radiant. | No. of Meteors. | Name. | References. | Weight. |
|---|---|---|---|---|---|---|
| 8 | 1928 Oct. 14·59c | 56°·0, + 11°·1 | 5 | Tau. | D. 49, No. 10; AMS 699, 251. | Good. |
| 9 | 15·65 | 84·0, + 10·0 | 7 | Ori. | D. 70, 6; AMS 104. | Good. |
| 10 | 16·64 | 51·6, − 1·4 | 6 | Eri. | D. 49, 11; AMS 248, 256. | Fair. |
| 11 | 16·64c | 56·8, + 6·4 | 4 | Tau. | D. 49, 13 and 14; AMS 251. | Good. |
| 12 | 16·64c | 80·6, − 13·2 | 6 | Lep. | ? D. 71, 3. | Fair. |
| 13 | 16·64 | 93·5, − 17·8 | 5 | CMa. | See No. 15; D. 78, 3. | Good. |
| 14 | 20·62c | 72·1, − 13·0 | 5 | Eri. | .. | Good. |
| 15 | 20·62 | 94·1, − 18·8 | 6 | CMa. | See No. 13; D. 78, 3. | Good. |

The letter c after the date indicates that observations on more than one night have been combined—a necessary step because I was not specially observing for these minor streams. Generally my radiants are determined from the observations of a single night. The table contains only those radiants which were derived from four or more meteors seen on one or two nights (except No. 11) and exhibiting a well-defined centre of radiation. In the "References" the prefix D indicates a radiant in Denning's General Catalogue, and AMS a radiant from the publications of the American Meteor Society.

With the southern radiants, of course, it is impossible in every case to find confirmatory radiants in the work of northern observers.

*Telescopic Meteors.*—On the suggestion of a New Zealand meteor observer, telescopic star charts of the region surrounding the Orionid radiant were prepared, with a view to sweeping the area for telescopic

meteors. The naked-eye results proved so interesting, however, that only on October 16, for fifteen minutes in a twilight sky, did I attempt a telescopic search, which did not yield any meteors. The other observer, Mr. Ll. L. Thomsen, experienced cloudy skies throughout the period. The experiment will be repeated during future showers by two or more observers in this country.

In conclusion, my observations have all been made with great care, and I am confident of their accuracy. While my radiants so clearly indicate the motion of the radiant-point, it must not be overlooked that experienced meteor observers in the northern hemisphere who have found a stationary radiant have been dealing with a type of meteor much more difficult to observe accurately than those that I saw in 1928 October. It is well known that a stationary radiant is contrary to theory, and it is therefore desirable that the problem of the Orionid shower should be settled. I have endeavoured to give my results as fully as possible, so that any meteor observer who has obtained different results will be in a position to understand the data on which I make the claim for a moving radiant for this shower.

It is significant that over one-quarter of the Orionids seen were of sufficient brightness to leave an image on a photographic plate. In coming years I hope to secure photographic evidence in support of the statements made in this paper, which may help to settle, at least as far as this shower is concerned, the vexed question of stationary radiation.

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*Photographies de la Comète Forbes* (1928*c*). By F. Quénisset.

(*Communicated by Dr. A. C. D. Crommelin.*)

Le 25 *Octobre* 1928, avant l'aube, à l'Observatoire de la station scientifique Suhalia près de Saint-Moritz (H^te^ Engadine, Suisse) situé à l'altitude de 2000 m., j'ai pris deux photographies de la Lumière Zodiacale à l'aide d'un objectif ultra-lumineux marque Rietzschel type " Prolinear " ouvert à $f/1{\cdot}9$, $f = 0^{m}{\cdot}135$. Le premier cliché a été exposé de $3^{h}\ 29^{m}$ à $3^{h}\ 55^{m}$ (T.U.), le second de $4^{h}\ 7^{m}$ à $4^{h}\ 20^{m}$. Plaques employées : Agfa marque " Andresa Ultra-Ortho " de sensibilité égale à 800 H. et D. L'atmosphère était, comme d'habitude dans cette haute région de l'Engadine, d'une transparence parfaite. Aussi les phototypes obtenus montrent admirablement la Lumière Zodiacale.

Pendant mon séjour dans cette station de montagne je m'étais proposé de faire quelques recherches photographiques de comètes et les clichés pris ce matin-là avaient été examinés dans ce but, mais malheureusement un peu trop rapidement et l'impression que la Lumière Zodiacale y avait laissée avait surtout retenu mon attention.

Tout dernièrement (un an après !), en relisant dans *Popular Astronomy* les notices si documentées de M. G. van Biesbroeck sur les comètes, j'eus connaissance de la position exacte de la comète Forbes observée

le 26 Octobre 1928 au Japon par M. Yamasaki et vis immédiatement que j'avais pris un cliché de cette même région de la constellation du Lion deux jours auparavant et qu'il était possible que la comète y eut laissé sa trace. Dès le premier examen je découvris, en effet, tout près de l'étoile B.D. 12° 2284 un petit objet nébuleux et le second cliché pris quelques minutes après confirma doublement la nature comètaire par son aspect et son déplacement. D'ailleurs sur un cliché de la même région obtenu deux mois plus tard, aucun objet ne figure à cette place.

Ce ne pouvait être que la comète Forbes, photographiée ainsi *deux fois* près d'un mois avant sa découverte par Forbes (le 21 Novembre 1928) et $1\frac{1}{2}$ jour avant la première observation de M. Yamasaki (le 26 Octobre 1928).

J'ai repéré ses positions sur un graphique tracé d'après les positions des étoiles données dans les Volumes L. et LIV. de la *Revised Harvard Photometry* et ai trouvé :

Le 25 Octobre 1928 à $3^h\ 42^m$ (T.U.) $\left\{\begin{matrix}\alpha = 10^h\ 55^m\ 30^s \\ \delta = +\ 11°\ 39'\end{matrix}\right.$

Le 25 Octobre 1928 à $4^h\ 13^m{\cdot}5$ (T.U.) $\left\{\begin{matrix}\alpha = 10^h\ 55^m\ 34^s \\ \delta = +\ 11°\ 37'\end{matrix}\right.$ éq. 1900·0.

Sur les photographies la comète apparait nettement comme une nébulosité de magnitude 7·5.

*Observatoire de Juvisy,*
*France.*

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## *Observations of the Partial Eclipse of the Sun at the Royal Observatory, Greenwich, 1929 November 1.*

(*Communicated by the Astronomer Royal.*)

At the beginning of the eclipse the sky was clear apart from fog. A series of five photographs were taken at short intervals after the commencement of the eclipse, with the 9-inch Thompson photographic refractor. The exposures were made from the object-glass end with the full aperture. A measurement of the photographs indicates that first contact was at

$$10^h\ 8^m\ 7^s{\cdot}8 \pm 0^s{\cdot}5 \text{ G.C.T.}$$

Observations made by noting, at five-second intervals, the positions of the cusps as seen on projection with the $6\frac{1}{2}$-inch Corbett refractor mounted on the 28-inch equatorial were less consistent and indicated a beginning $3^s$ later.

The conditions at the end of the eclipse were not so good, but from three photographs the last contact was deduced as

$$11^h\ 56^m\ 1^s \pm 5^s.$$

From short exposures with the astrographic telescope, the last contact was deduced as

$$11^h\ 56^m\ 2^s \pm 10^s.$$

From the elements given in the *Nautical Almanac* Dr. L. J. Comrie has computed the times of first and last contact as

$$10^h\ 7^m\ 56^s$$
and $$11\ 55\ 47.$$

If the longitude of the sun is increased by $1''{\cdot}5$ and that of the moon is decreased by $0''{\cdot}5$—corrections which appear from other sources to be nearly correct—the computed times are

$$10^h\ 8^m\ 3^s$$
and $$11\ 55\ 51.$$